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HANZHE ZHANG

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To Mom and Dad

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I vaguely remembered on my first day in Chicago listening to an old guy babbling about women and college. Who would have thought I wrote a dissertation exactly on this topic under this “old guy’s” tutelage? I did not try to follow Gary Becker: it just so happens what I thought was a totally original painting was simply a clumsy recoloring of his masterful sketch. Professor Becker: One year since you have passed, I still miss you and feel helpless at times. I no longer have the luxury with my future projects to simply tell others: “No *you* didn’t get it. Professor Becker said it is important.” I hope I can carry forward a glimmer of your fire as your last student.

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In Chapter 2, I show that people have incentives to take seemingly risky gambles before they enter a competitive matching market, and derive implications about occupational choices and wealth inequality. I am very grateful to the discussions with and the suggestions from Gary Becker, Pierre-André Chiappori, Alex Frankel, Thomas Gall, John Hatfield, Scott Kominers, Phil Reny, Larry Samuelson, Lones Smith, Hugo Sonnenschein, Balázs Szentes, Richard Van Weelden, and Alessandra Voena. Discussions at various University of Chicago seminars, twelfth Social Choice and Welfare Conference in Boston, 25th Jerusalem School in Economic Theory, and International Workshop on Game Theory and Economic Applications in São Paulo are also greatly appreciated.

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Chapter 1 Human Capital Investments and the Marriage Market

Abstract: I provide a dynamic equilibrium framework to simultaneously study human capital investments and the marriage market. A set of variables is endogenously determined: investments, income, marriage-age, marriage matching, and the division of marriage surplus. I prove equilibrium existence with Glicksberg and Arzelà-Ascoli theorems and show the wide applicability of the techniques to other papers. I provide a marriage-market-based explanation to the college gender gap puzzle: more women than men go to college although women's tradeoff between career and fertility lowers their incentive for college. I also derive a set of other new implications including the relationships between marriage-age and income.

Keywords: investment-and-matching models, equilibrium existence, the college gender gap puzzle, marriage-age patterns

JEL: C78, D10, J12

1.1 Introduction

Although human capital investments (Becker, 1964) and the marriage market (Becker, 1991) have been extensively studied, they have rarely been studied simultaneously. Yet, an individual's investments and marriage decisions depend on each other as well as other people's decisions. To fill this gap, I construct in this paper a dynamic equilibrium framework to derive a set of new implications about the labor market and the marriage market.

A set of variables is endogenously determined: college and career investment decisions, distributions of income and marriage-age, and the matching and the division of surplus in the marriage market. The equilibrium is easily characterized with cutoff strategies and non-assortative matching. The main technical difficulty is to prove equilibrium existence. I apply Glicksberg's fixed-point theorem and the Arzelà-Ascoli theorem to tackle this difficulty and show the wide applicability of the techniques to other papers in the literature. Previous existence proofs have hinged on such restrictions as the single dimensionality of the marriage characteristics, discrete support of the marriage characteristics, and supermodularity of the marriage surplus function. The current approach renders these restrictions unnecessary.

The framework explains in a unified manner a set of observations in the labor market and the marriage market, with the simultaneous determination of many important variables and the key assumption of gender difference in reproductive fitness: women are more likely to become reproductively fit after thirty[1]. First, I provide a new marriage-market-based explanation to the college gender gap puzzle, i.e., more women than men go to college although their tradeoff between career and fertility supposedly lowers their incentive for college (Section 1.5.1). Then, I explain the gender-specific, changing, non-monotonic relationships between marriage-age and personal income (Sections 1.5.2 and 1.5.3), and the changing rela-

[1]The assumption is based on the biological fact that the risk of poor birth outcome increases significantly with mother's age. For example, according to the U.S. Department of Health and Human Services, about one-third of couples in which the wife is over thirty-five have encountered fertility problems.

tionships between a woman's marriage-age and her husband's income (Section 1.5.4). Finally, I discover a new marriage-delaying factor: people delay marriage when they are uncertain about their labor market prospects (Section 1.5.5).

The paper contributes to the theoretical literature as well as several strands of the applied literature. The theoretical framework adds repeated investments with uncertain returns to previous papers that only consider agents who live for two periods and make investments with deterministic returns (Iyigun and Walsh, 2007; Chiappori et al., 2009; Low, 2015; Chiappori et al., 2015b), and, more importantly, introduces new equilibrium existence techniques applicable to all the previous papers. Whereas a sizable theoretical literature focuses on the social efficiency of the premarital investments (Cole et al., 2001b,a; Peters and Siow, 2002; Iyigun and Walsh, 2007; Chiappori et al., 2009; Dizdar, 2013; Hatfield et al., 2014; Nöldeke and Samuelson, 2014), this paper provides a unifying theory to address a set of applied questions. First, this paper offers a new marriage-market-based explanation of the college gender gap puzzle (Goldin et al., 2006; Mulligan and Rubinstein, 2008; Becker et al., 2010b,a; **?**) and calls for more attention to investigate the marriage market returns to college (Chiappori et al., 2009, 2015a; Bruze, 2015). Second, the paper fully explains the gender-specific changing relationships between marriage-age and personal income, unifying and extending the previous literature (Becker, 1973; Keeley, 1974, 1977, 1979; Bergstrom and Bagnoli, 1993; Bergstrom and Schoeni, 1996). Finally, the paper stresses the biological gender difference in reproductive fitness and examines the changing marriage prospects of highly educated women, an issue of increasing importance as more women progress in the labor market but delay marriage and childbearing (Siow, 1998; Low, 2015; Bertrand et al., 2010; Chiappori et al., 2012; Díaz-Giménez and Giolito, 2013; Goldin, 2014).

The rest of the paper is organized as follows. Section 1.2 sets up the general theoretical framework. Section 1.3 characterizes the equilibrium and proves equilibrium existence. Section 1.4 presents a simplified version of the framework and proves equilibrium uniqueness. Section 1.5 derives implications of the model about the labor market and the marriage market. Section 1.6 concludes. Appendix 1.7 presents proofs of lemmas, theorems and propositions. Appendix 1.8 describes data and includes additional figures.

1.2 The General Theoretical Framework

My analysis is based on the following framework. Each period, a new generation of men and women reach adulthood. They have heterogenous income-earning abilities and decide whether or not to go to college. A college investment improves a person's income with uncertainty. Depending on the income prospects of the job after college, a college graduate can make what I call a career investment to change the income prospects. Examples of a career investment include choosing a different career, searching for a new job, or additional schooling. People marry when they finish investments. Each period, overlapping generations of men and women participate in a frictionless transferable-utilities marriage market, and match and bargain over the division of their marriage surplus until no one can benefit from choosing a different partner; the marriage market reaches a stable outcome. A couple's marriage surplus depends on the husband's income, the wife's income, and her reproductive fitness. Whereas men stay reproductively fit, women may become less reproductively fit in their thirties. The reproductive fitness is the only gender difference in our analysis.

Let me formally introduce the theoretical framework. There are an infinite number of discrete periods. At the beginning of each period, unit masses of men and women reach adulthood and become eligible for the labor market and the marriage market. They are endowed with heterogeneous income-earning abilities indexed by θ_m and θ_w. The abilities are distributed according to continuous and strictly increasing distributions F_m and F_w on $[0, 1]$. Agents each live for three periods, which can be thought as ages 20-25, 25-30, and 30-35. They are risk-neutral and discount each period by δ. The agents derive utilities from the labor market income Y and the marriage market payoff V, and dis-utilities from any costs C paid for human capital investments.

1.2.1 College and Career Investments

Each age 1 agent chooses whether or not to make a *college investment*. An agent who does not go to college enters the labor market and the marriage market, and realizes an income Y_i drawn from distribution P_{i1}, $i = m, w$. The income distributions can have continuous or discrete supports. An agent who goes to college pays a cost C_{i1}, and delays entering the labor market and the marriage market. Assume for now that it is infeasible to invest and to enter the labor market and the marriage market in the same period.[2]

Each age 2 θ_i-ability college graduate receives an income Y_i offer drawn from distribution $P_{i2}(\cdot|\theta_i)$. Assume that higher-ability agents receive better income offers; for $i = m, w$, $P_{i2}(\cdot|\theta_i)$ strictly first-order stochastically dominates $P_{i2}(\cdot|\theta_i')$ if $\theta_i > \theta_i'$. The agent decides whether or not to accept the income offer. An agent who accepts the income offer earns lifetime income Y_i and enters the marriage market. An agent who rejects the income offer makes a *career investment* costing C_{i2}, and skips the labor market and the marriage market in the current period. The career investment can be working hard at the current job, spending more effort searching for a better job offer, receiving more training, or additional schooling.

An age 3 θ_i-ability agent who has made the career investment receives an income offer Y drawn from distribution $P_{i3}(\cdot|\theta_i)$. Assume that higher ability agents receive better income offers from career investments; for $i = m, w$, $P_{i3}(\cdot|\theta_i)$ strictly first-order stochastically dominates $P_{i3}(\cdot|\theta_i')$ if $\theta_i > \theta_i'$. An agent at this point has no choice but to enter the labor market and the marriage market, and earns a lifetime income Y_i.

Let $\sigma_{i1}(\theta_i)$ represent the probability a θ_i-ability agent makes a college investment and

[2]Section 1.5.5 relaxes this assumption and shows that investing and entering the marriage market in the same period is dominated by investing and delaying marriage for a sufficiently patient agent.

$\sigma_{i2}(\theta_i, Y_{i2})$ the probability a θ_i-ability agent makes a career investment after receiving an income Y_{i2} offer. Let $\sigma_m = (\sigma_{m1}(\cdot), \sigma_{m2}(\cdot,\cdot))$ and $\sigma_w = (\sigma_{w1}(\cdot), \theta_{w1}(\cdot,\cdot))$ respectively denote men's and women's stationary and ability-symmetric strategies.

1.2.2 The Marriage Market

Whereas men of all ages and women who enter the marriage market at age 1 and age 2 stay reproductively fit ($R = R_H$), women who enter the marriage market at age 3 may have lower reproductive fitness ($R \leq R_H$), distributed according to distribution Φ. The marriage surplus $S(Y_m, Y_w, R)$ a couple generates depends on the husband's income Y_m, the wife's income Y_w, and her reproductive fitness R. Assume that S is non-negative, twice differentiable, and strictly increasing in every argument, and that the singles' marriage surplus is zero.

Because agents make different investment decisions and enter the marriage market at different ages, there are overlapping generations of men and women in the marriage market in each period. The stationary and symmetric strategies σ_m and σ_w *induce* stationary measures μ_m on $[Y_{mL}, Y_{mH}]$ and μ_w on $[Y_{wL}, Y_{wH}] \times [R_L, R_H]$ which together describe the marriage market.

Men's income measure μ_m is induced as follows. The men who earn less than income Y_m include those who do not go to college and marry at age 1 with an income less than Y_m, those who accept an offer of income less than Y_m and marry at age 2, and those who reject an offer after college and receive an income less than Y_m at age 3,

$$\begin{aligned}\mu_m([Y_{mL}, Y_m]) \;=\; & \left[\int (1-\sigma_{m1}(\theta_m))dF_m(\theta_m)\right] P_{m1}(Y_m) + \\ & \int \int_{Y_{mL}}^{Y_m} \left[1-\sigma_{m2}(\theta_m, Y_{m2})\right] dP_{m2}(Y_{m2}|\theta_m)dF_m(\theta_m) + \\ & \int \int \sigma_{m2}(\theta_m, Y_{m2}) P_{m3}(Y_m|\theta_m) dP_{m2}(Y_{m2}|\theta_m)dF_m(\theta_m).\end{aligned}$$

Women's marriage characteristics measure μ_w is similarly characterized. The women who earn less than income Y_w and stay reproductively fit include those who do not go to college and marry at age 1 with income less than Y_m, those who go to college and accept an income offer less than Y_w at age 2, and those who make the career investment stay reproductively fit and receive an income less than Y_w in the third period,

$$\begin{aligned}\mu_w([Y_{wL}, Y_w] \times R_H) &= \left[\int (1 - \sigma_{w1}(\theta_w)) dF_w(\theta_w)\right] P_{w1}(Y_w) + \\ &\int \int_{Y_{wL}}^{Y_w} [1 - \sigma_{w2}(\theta_w, Y_{w2})]\, dP_{w2}(Y_{w2}|\theta_w) dF_w(\theta_w) + \\ &\int \int \sigma_{w2}(\theta_w, Y_{w2}) P_{w3}(Y_m|\theta_m) dP_{w2}(Y_{w2}|\theta_w) dF_w(\theta_w)\phi(R_H)\end{aligned}$$

where $\phi(R_H)$ is the probability measure of being highly reproductive. All of $R < R_H$ women have entered the marriage market at age 3,

$$\mu_w([Y_{wL}, Y_w] \times [R_L, R]) = \int \int \sigma_{w2}(\theta_w, Y_{w2}) P_{w3}(Y_w|\theta_w) dP_{w2}(Y_{w2}|\theta_w) dF_w(\theta_w)\Phi(R).$$

In a marriage market described by (μ_m, μ_w), men and women match and bargain over their marriage surplus. A *stable outcome* of the marriage market (μ_m, μ_w) consists of a *matching* described by a measure μ on $[Y_{mL}, Y_{mH}] \times [Y_{wL}, Y_{wH}] \times [R_L, R_H]$ with marginals μ_m and μ_w, and *marriage payoff functions* $V_m : [Y_{mL}, Y_{mH}] \to \mathbb{R}$ and $V_w : [Y_{wL}, Y_{wH}] \times [R_L, R_H] \to \mathbb{R}$. $\mu(\mathcal{Y}_m \times \mathcal{Y}_w \times \mathcal{R})$ denotes the mass of matches between men with characteristics $Y_m \in \mathcal{Y}_m$ and women with characteristics $(Y_w, R) \in \mathcal{Y}_w \times \mathcal{R}$. Stable marriage payoff functions V_m and V_w satisfy the following stability conditions. First, every agent receives weakly more than being single: $V_m(Y_m) \geq 0$ for all Y_m and $V_w(Y_w, R) \geq 0$ for all (Y_w, R). Second, every matched couple divides the entire surplus: $V_m(Y_m) + V_w(Y_w, R) = S(Y_m, Y_w, R)$ for all $(Y_m, Y_w, R) \in \text{supp}(\mu)$. Third, no unmatched couple can feasibly prefer to match with each other: $V_m(Y_m) + V_w(Y_w, R) \geq S(Y_m, Y_w, R)$ for all $Y_m \in \text{supp}(\mu_m)$ and $(Y_w, R) \in \text{supp}(\mu_w)$. In words, in a stable outcome, no one is willing to leave his or her partner and form a new matching with someone else, because the current marriage payoff cannot be improved upon.

For any marriage market (μ_m, μ_w), a stable outcome (μ, V_m, V_w) exists (Gretsky et al., 1992). For any stable matching μ, let $X_m(Y_w, R) = \{Y_m | (Y_m, Y_w, R) \in \text{supp}(\mu)\}$ denote the set of men with which a (Y_w, R) woman is matched, and $X_w(Y_m) = \{(Y_w, R) | (Y_m, Y_w, R) \in \text{supp}(\mu)\}$ the set of women with which a Y_m man is matched. The stability conditions imply that for any Y_m,

$$V_m(Y_m) = S(Y_m, Y_w, R) - V_w(Y_w, R) \quad \forall (Y_w, R) \in X_m(Y_m)$$

and

$$V_m(Y_m) \geq S(Y_m, Y_w, R) - V_w(Y_w, R) \quad \forall (Y_w, R) \in \text{supp}(\mu_w).$$

The two conditions together imply

$$V_m(Y_m) = \max_{(Y_w, R) \in \text{supp}(\mu_w)} \left[S(Y_m, Y_w, R) - V_w(Y_w, R) \right].$$

In words, the partner with which any agent is matched gives the agent the highest payoff possible. The similar condition holds for women,

$$V_w(Y_w, R) = \max_{Y_m \in \text{supp}(\mu_m)} \left[S(Y_m, Y_w, R) - V_m(Y_m) \right].$$

The stability conditions above do not restrict $V_m(Y_m)$ or $V_w(Y_w, R)$ for out-of-support marriage characteristics. We make the assumption that agents have rational expectations, and know the stable marriage payoffs and partners if they enter the marriage market with out-of-support marriage characteristics. Define for $Y_m \notin \text{supp}(\mu_m)$,

$$V_m(Y_m) \equiv \max_{(Y_w, R) \in \text{supp}(\mu_w)} [S(Y_m, Y_w, R) - V_w(Y_w, R)],$$

$$X_w(Y_m) \equiv \operatorname*{argmax}_{(Y_w, R) \in \text{supp}(\mu_w)} [S(Y_m, Y_w, R) - V_w(Y_w, R)],$$

and for $(Y_w, R) \notin \text{supp}(\mu_w)$,

$$V_w(Y_w, R) \equiv \max_{Y_m \in \text{supp}(\mu_m)} [S(Y_m, Y_w, R) - V_m(Y_m)],$$

$$X_m(Y_w, R) \equiv \operatorname*{argmax}_{Y_m \in \text{supp}(\mu_m)} [S(Y_m, Y_w, R) - V_m(Y_m)].$$

1.3 Equilibrium

An equilibrium consists of men's and women's investment strategies σ^*_m and σ^*_w, distributions of marriage characteristics μ^*_m and μ^*_w, a matching μ^*, and marriage payoff functions V^*_m and V^*_w. In equilibrium, agents choose the investments that maximize their utilities, the utility-maximizing investments induce distributions of income and reproductive fitness in the marriage market, and the matching and the marriage payoff functions on which the agents base their decisions form a stable outcome of the induced marriage market.

Definition 1. $(\sigma^*_m, \sigma^*_w, \mu^*_m, \mu^*_w, \mu^*, V^*_m, V^*_w)$ is an *equilibrium* if

1. Investment strategy $\sigma^*_m(\theta_m) = (\sigma^*_{m1}(\theta_m), \sigma^*_{m2}(\theta_m, \cdot))$ maximizes each θ_m-ability man's expected utility when men's marriage payoff function is V^*_m, and the investment strategy $\sigma^*_w(\theta_w) = (\sigma^*_{w1}(\theta_m), \sigma^*_{w2}(\theta_m, \cdot))$ maximizes each θ_w-ability woman's expected utility when women's marriage payoff function is V^*_w,

2. Men's investment strategy σ^*_m induces men's marriage characteristics measure μ^*_m, and women's investment strategy σ^*_w induces women's marriage characteristics measure μ^*_w, and

3. (μ^*, V^*_m, V^*_w) is a stable outcome of the marriage market (μ^*_m, μ^*_w).

I characterize equilibrium components in the following order: investments (Section 1.3.1), distributions of marriage characteristics (Section 1.3.2), matching (Section 1.3.3), and marriage payoff functions (Section 1.3.4). Finally, I prove equilibrium existence (Section 1.3.5).

1.3.1 College and Career Investments

Suppose that agents make their investment decisions with respect to stable marriage payoff functions V_m and V_w. The marriage payoff functions are continuous and strictly increasing in marriage characteristics, in particular in incomes and reproductive fitness (Gretsky et al., 1992). Because the marriage payoff functions are increasing in incomes and a higher-ability agent receives better income offers than a lower-ability agent making the same investment in the first-order stochastic sense, a higher-ability agent receives a higher expected marriage payoff from an investment than a lower-ability agent who makes the same investment. Therefore, an agent makes a college investment if and only if his or her ability is sufficiently high, and makes a career investment if and only if the income sufficiently low. Simple ability cutoffs θ_{m1} and θ_{w1} characterize optimal college investment strategies and reservation incomes $Y_{m2}(\theta_m)$ and $Y_{w2}(\theta_w)$ characterize optimal career investment strategies: for all θ_i and Y_{i2}, $i = m, w$,

$$\sigma_{i1}^*(\theta_i) = \begin{cases} 1 & \theta_i \geq \theta_{i1} \\ 0 & \theta_i < \theta_{i1} \end{cases}, \quad \sigma_{i2}^*(\theta_i, Y_{i2}) = \begin{cases} 0 & Y_{i2} \geq Y_{i2}(\theta_i) \\ 1 & Y_{i2} < Y_{i2}(\theta_i) \end{cases}.$$

1.3.1.1 Men's Investments

By the Principle of Optimality, men's optimal investment strategy $\sigma_m^* = (\sigma_{m1}^*, \sigma_{m2}^*)$ can be solved by backward induction. We first solve men's optimal career investment strategy σ_{m2}^*. Suppose that an age 2 θ_m-ability man who has made the college investment receives an income offer Y_{m2}. If he accepts the offer, his utility is

$$Y_{m2} + V_m(Y_{m2}).$$

If he rejects the offer and makes a career investment, his utility is

$$-C_{m2} + \delta\mathbb{E}[Y_{m3} + V_m(Y_{m3})|\theta_m].$$

Therefore, he makes a career investment if and only if

$$-C_{m2} + \delta\mathbb{E}[Y_{m3} + V_m(Y_{m3})|\theta_m] \geq Y_{m2} + V_m(Y_{m2}).$$

With rearrangement,

$$\underbrace{\delta\left[\mathbb{E}(Y_{m3}|\theta_m) - Y_{m2}\right]}_{\text{labor market gain}} + \underbrace{\delta\left[\mathbb{E}(V_m(Y_{m3})|\theta_m) - V_m(Y_{m2})\right]}_{\text{marriage market gain}} \geq \underbrace{(1-\delta)\left[Y_{m2} + V_m(Y_{m2})\right]}_{\text{cost of delay}} + \underbrace{C_{m2}}_{\text{investment cost}}$$

where the left-hand side represents the benefits of the labor market gain and the marriage market gain, and the right-hand side represents the cost of delay and the cost of investment. There is a reservation income $Y_{m2}(\theta_m)$ for each θ_m such that an ability θ_m agent only accepts an income offer exceeding $Y_{m2}(\theta_m)$. $Y_{m2}(\theta_m)$ satisfies

$$-C_{m2} + \delta\mathbb{E}[Y_{m3} + V_m(Y_{m3})|\theta_m] = Y_{m2}(\theta_m) + V_m(Y_{m2}(\theta_m)).$$

$Y_{m2}(\theta_m) \in (-\infty, Y_{mH}]$ is unique and is increasing in θ_m since $\delta\mathbb{E}[Y_{m3} + V_m(Y_{m3})|\theta_m]$ and $V_m(\cdot)$ are continuous and increasing in θ_m.

We can solve for the optimal college investment strategy given the optimal career investment strategy. A θ_m-ability man who does not go to college gets expected utility

$$\mathbb{E}\left[Y_{m1} + V_m(Y_{m1})\right].$$

If he goes to college, his expected utility is

$$-C_{m1} + \delta\mathbb{E}\left[\max\left\{Y_{m2} + V_m(Y_{m2}), -C_{m2} + \delta\mathbb{E}[Y_{m3} + V_m(Y_{m3})|\theta_m]\right\}|\theta_m\right].$$

The expected utility of the college investment increases with θ_m, because P_{m2} and P_{m3} are first-order stochastically ranked by θ_m. Therefore, an agent goes to college only if he is above ability θ_{m1}, the smallest solution to

$$
\begin{aligned}
&-C_{m1} + \delta\mathbb{E}[\max\{Y_{m2} + V_m(Y_{m2}), \\
&-C_{m2} + \delta\mathbb{E}[Y_{m3} + V_m(Y_{m3})|\theta_{m1}]|\theta_{m1}\}] \quad \geq \quad \mathbb{E}\left[Y_{m1} + V_m(Y_{m1})\right].
\end{aligned}
$$

Since the expected utility of going to college strictly increases with ability, the cutoff θ_{m1} is unique. Without any restrictions on the parameters, it is possible that every man invests or no man invests ($\theta_{m1} = 0$ or $\theta_{m1} = 1$).

1.3.1.2 Women's Investments

Women's optimal investment strategy can also be solved by backward induction. Suppose that a θ_w-ability woman receives an income offer of Y_{w2}. If she accepts the offer, her utility is

$$Y_{w2} + V_w(Y_{w2}, R_H).$$

If she rejects the offer and makes a career investment, her utility is

$$-C_{w2} + \delta\mathbb{E}_{Y_{w3},R}[Y_{w3} + V_w(Y_{w3}, R)|\theta_w].$$

She makes a career investment if and only if

$$
\begin{aligned}
&\underbrace{\delta[\mathbb{E}(Y_{w3}|\theta_w) - Y_2]}_{\text{labor market gain}} + \underbrace{\delta[\mathbb{E}(V_w(Y_{w3}, R_H)|\theta_w) - V_w(Y_{w2}, R_H)]}_{\text{marriage market gain}} \\
\geq \; &\underbrace{(1-\delta)[Y_{w2} + V_w(Y_{w2}, R_H)]}_{\text{cost of delay}} + \underbrace{C_{w2}}_{\text{investment cost}} + \underbrace{\delta\mathbb{E}[V_w(Y_{w3}, R_H) - V_w(Y_{w3}, R)|\theta_w]}_{\text{fitness cost}}.
\end{aligned}
$$

Women face a fitness cost in addition to the same benefits and costs men face. A woman expects to gain in the income dimension of the marriage payoff with a career investment but expects to lose in the fitness dimension. A θ_w-ability woman's optimal career investment strategy is characterized by a reservation income $Y_{w2}(\theta_w)$, which is the unique solution to

$$Y_{w2}(\theta_w) + V_w(Y_{w2}(\theta_w), R_H) = -C_{w2} + \delta \mathbb{E}_{Y_{w3},R}[Y_{w3} + V_w(Y_{w3}, R)|\theta_w].$$

Given the optimal career investment, we can then solve the optimal college investment. A θ_w-ability woman who does not go to college gets

$$\mathbb{E}[Y_{w1} + V_w(Y_{w1}, R_H)].$$

A θ_w-ability woman who goes to college gets

$$-C_{w1} + \delta \mathbb{E}[\max\{Y_{w2} + V_w(Y_{w2}, R_H), -C_{w2} + \delta \mathbb{E}[Y_{w3} + V_w(Y_{w3}, R)|\theta_w]\}|\theta_w].$$

All women above cutoff ability θ_{w1} go to college, where $\theta_{w1} \in [0, 1]$ is the smallest solution to

$$\begin{aligned} -C_{w1} + \delta \mathbb{E}[\max\{Y_{w2} + V_w(Y_{w2}, R_H), \\ -C_{w2} + \delta \mathbb{E}[Y_{w3} + V_w(Y_{w3}, R)|\theta_{w1}]\}|\theta_{w1}] \quad &\geq \quad \mathbb{E}[Y_{w1} + V_w(Y_{w1}, R_H)]. \end{aligned}$$

1.3.2 Distributions of Marriage Characteristics

Equal unit masses of men and women are in the marriage market. Men above ability θ_{m1} go to college, and among them, those who receive an income offer above $Y_{m2}(\theta_m)$ accept the offer and enter the marriage market at age 2, and those who receive an income offer below $Y_{m2}(\theta_m)$ make the career investment and enter the marriage market at age 3. The measure of men's marriage characteristics induced by the optimal strategy is

$$\begin{aligned} \mu_m([Y_{mL}, Y_m]) \quad = \quad & F_m(\theta_{m1}) P_{m1}(Y_m) + \\ & \int_{\theta_{m1}}^{1} P_{m2}(\min\{Y_m, Y_{m2}(\theta_m)\}|\theta_m) dF_m(\theta_m) + \\ & \int_{\theta_{m1}}^{1} [1 - P_{m2}(Y_{m2}(\theta_m)|\theta_m)] P_{m3}(Y_m|\theta_m) dF_m(\theta_m). \end{aligned}$$

The measure of women's characteristics is similarly derived.

$$\begin{aligned}\mu_w([Y_{wL}, Y_w] \times R_H) &= F_w(\theta_{w1})P_{w1}(Y_w) + \\ &\int_{\theta_{m1}}^{1} P_{w2}(\min\{Y_w, Y_{w2}(\theta_w)\}|\theta_w)dF_w(\theta_w) + \\ &\int_{\theta_{w1}}^{1} [1 - P_{w2}(Y_{w2}(\theta_w)|\theta_w)]P_{w3}(Y_w|\theta_w)dF_w(\theta_w)\phi(R_H).\end{aligned}$$

For $R < R_H$,

$$\mu_w([Y_{wL}, Y_w] \times [R_L, R]) = \int_{\theta_{w1}}^{1} [1 - P_{w2}(Y_{w2}(\theta_w)|\theta_w)]P_{w3}(Y_w|\theta_w)dF_w(\theta_w)\Phi(R).$$

1.3.3 Marriage Matching

By Gretsky et al. (1992), stable matching μ exists and solves the primal linear program PP(μ_m, μ_w)

$$\max_{\widetilde{\mu}} \int S d\widetilde{\mu} \quad \text{s.t. } \widetilde{\mu} \text{ has marginals } \mu_m \text{ and } \mu_w.$$

That is, stable matching maximizes the total surplus of the marriage market (μ_m, μ_w). To further characterize the stable matching pattern, we need to impose additional restrictions on the marriage surplus function.

Lemma 1. *Suppose that the marriage surplus function is strictly supermodular (submodular) in incomes, that is, for all $Y'_m > Y_m$ and $Y'_w > Y_w$, $S(Y'_m, Y'_w, R) + S(Y_m, Y_w, R) > (<) S(Y'_m, Y_w, R) + S(Y_m, Y'_w, R)$. Men and women of a fixed fitness level are positive-assortatively (negative-assortatively) matched: for almost all (Y'_m, Y'_w, R) and (Y_m, Y_w, R) in the support of μ, $Y'_m > Y_m$ if and only if $Y'_w > (<) Y_w$.*

Let me characterize the stable matching with continuous incomes and two fitness levels R_L and R_H and the surplus function strictly supermodular in Y_m and Y_w and in Y_m and R.[3]

[3] See Chiappori et al. (2012) and Low (2015) for similar matching models with a continuous income characteristic and a second binary characteristic.

The key complication compared to the basic one-dimensional setting of Becker (1973) is that agents of identical characteristics may be matched with partners of different marriage characteristics. In this particular setting, a man may be indifferent between a high-income less-fit woman and a low-income fit woman. Top-income men are matched to top-income fit women, and bottom-income men are matched to bottom-income fit women. The complication is that some middle-income men may be matched to a lower-income fit woman with or a higher-income less-fit woman. The matching can be divided into two scenarios depending on whether or not there exists a positive mass of the middle-income men. The importance of fitness plays a key role in the distinction of the two cases. When fitness is an important consideration in marriage, less-fit women cannot compensate their lower fitness with higher income to marry a top-income man. In this case, higher-income men marry fit women and lower-income men marry less-fit women. When the fitness is less important so that a woman can compensate her fitness loss with income gain to attract a top-income man, a positive mass of middle-income men is matched to either a high-income less-fit woman or a low-income fit woman. Figure 1.1 illustrates the two cases of equilibrium matching. The exact condition that formalizes the importance of the fitness and the stable matching patterns are delineated in the following lemma.

Lemma 2. *Suppose that $S(Y_m, Y_w, R)$ is twice differentiable, and strictly supermodular in Y_m and Y_w, and in Y_m and R, and that the marriage market is described by marriage characteristics measures μ_m with support on $[Y_{mL}, Y_{mH}]$ and μ_w with support on $[Y_{wL}, Y_{wH}] \times \{R_L, R_H\}$. Define income distributions $G_m(Y_m) = \mu_m([Y_{mL}, Y_m])$, $G_{wL}(Y_w) = \mu_w([Y_{wL}, Y_w] \times R_L)$ and $G_{wH}(Y_w) = \mu_w([Y_{wL}, Y_w] \times R_H)$, and cutoff income $Y_m^* = G_m^{-1}(G_{wL}(Y_{wH}))$. Consider the following condition*

$$
\begin{aligned}
& S(Y_m^*, G_{wH}^{-1}(G_m(Y_m) - G_{wL}(Y_{wH})), R_H) + S(Y_m^*, Y_{wH}, R_L) \\
\geq \; & S(Y_m^*, G_{wH}^{-1}(G_m(Y_m) - G_{wL}(Y_{wH})), R_H) + S(Y_m, Y_{wH}, R_L) \quad \forall Y_m \geq Y_m^* \qquad (1.1)
\end{aligned}
$$

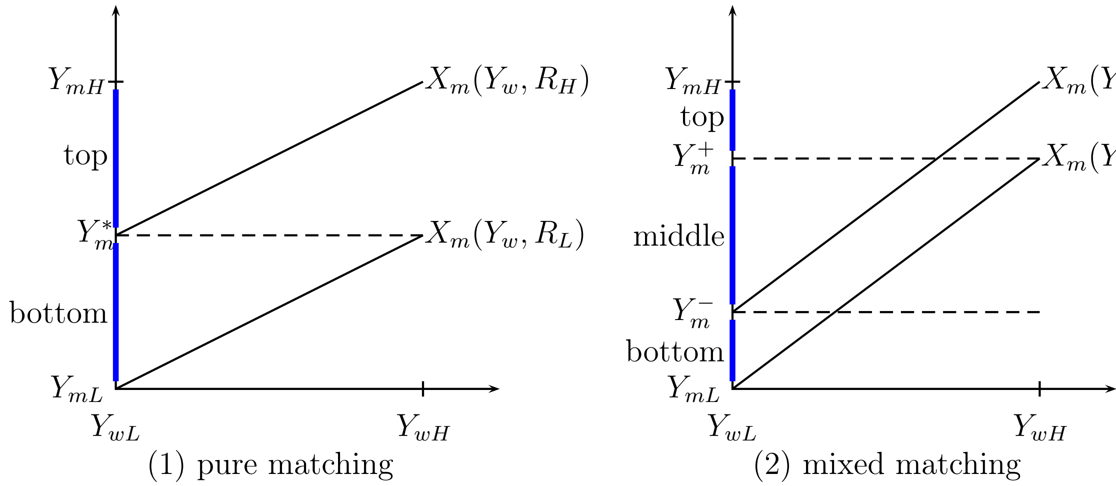

Figure 1.1: Two stable matching patterns. The left panel illustrates the pure matching that arises when fitness is important. The right panel illustrates the mixed matching that arises when fitness is less important.

1. *(Pure Matching) If and only if Condition 1.1 holds, almost all income $Y_m \geq Y_m^*$ men marry R_H women and almost all income $Y_m \leq Y_m^*$ men marry R_L women.*

2. *(Mixed Matching) If and only if Condition 1.1 does not hold, there exist Y_m^- and Y_m^+ such that almost all income $Y_m \geq Y_m^+$ men marry R_H women, almost all income $Y_m \leq Y_m^-$ men marry R_L women, and almost all income $Y_m \in [Y_m^-, Y_m^+]$ men marry either a R_L or R_H woman with positive probability.*

Condition 1.1 is the "twisted-buyer" condition (Chiappori et al., 2010) specialized to the setting with binary fitness. The general stable matching pattern is more complicated to characterize and depends on more assumptions about the distributions of marriage characteristics as well as the marriage surplus function.

1.3.4 Marriage Payoffs

By Gretsky et al. (1992), stable marriage payoff functions V_m and V_w exist and solve the dual linear problem DP(μ_m, μ_w)

$$\min_{\widetilde{V}_m, \widetilde{V}_w} \int \widetilde{V}_m d\mu_m + \int \widetilde{V}_w d\mu_w$$

$$\text{s.t.} \quad \widetilde{V}_m(Y_m) \geq 0, \widetilde{V}_w(Y_w, R) \geq 0, \widetilde{V}_m(Y_m) + \widetilde{V}_w(Y_w, R) \geq S(Y_m, Y_w, R) \, \forall (Y_m, Y_w, R).$$

Furthermore, if the marriage surplus function is twice differentiable and strictly increasing, then stable marriage payoff functions are twice differentiable and strictly increasing. A person is matched to the partner who gives him or her the highest possible marriage payoff,

$$V_w(Y_w, R) = \max_{Y_m \in \text{supp}(\mu_m)} [S(Y_m, Y_w, R) - V_m(Y_m)].$$

By the Envelope Theorem, for any (Y_w, R) and $Y_m \in X_m(Y_w, R)$,

$$\frac{\partial V_w(Y_w, R)}{\partial Y_w} = S_2(Y_m, Y_w, R).$$

By integration, women's stable marriage payoff function satisfies

$$V_w(Y_w, R) - V_w(Y_{wL}, R) = \int_{Y_{wL}}^{Y_w} S_2(\widetilde{Y}_m, \widetilde{Y}_w, R) d\widetilde{Y}_w \quad \widetilde{Y}_m \in X_m(\widetilde{Y}_w, R).$$

Similarly, men's stable marriage payoff function satisfies[4]

$$V_m(Y_m) - V_m(Y_{mL}) = \int_{Y_{mL}}^{Y_m} S_1(\widetilde{Y}_m, \widetilde{Y}_w, \widetilde{R}) \, dR \, d\widetilde{Y}_m \quad (\widetilde{Y}_w, \widetilde{R}) \in X_w(\widetilde{Y}_m).$$

1.3.5 Equilibrium Existence

Theorem 1. *An equilibrium* $(\sigma_m^*, \sigma_w^*, \mu_m^*, \mu_w^*, \mu^*, V_m^*, V_w^*)$ *exists.*

In order to establish equilibrium existence, first fix a pair of marriage payoff functions (V_m, V_w). We have characterized the optimal investment strategy $(\sigma_m, \sigma_w) = \Gamma_\sigma(V_m, V_w)$ with respect to (V_m, V_w) in Section 1.3.1, the pair of marriage characteristics measures $(\mu_m, \mu_w) = \Gamma_\mu(\Gamma_\sigma(V_m, V_w))$ induced by the optimal strategy $\Gamma_\sigma(V_m, V_w)$ in Section 1.3.2, and the set of stable marriage payoff functions $\Gamma_V(\Gamma_\mu(\Gamma_\sigma(V_m, V_w)))$ of the marriage market $\Gamma_\mu(\Gamma_\sigma(V_m,$

[4] In order for the above equations to be well-defined, we must have $S_1(Y_m, Y_w, R)$ to be equal for all $(Y_w, R) \in X_w(Y_m)$ for any given Y_m. This is warranted because for $(Y_w, R), (Y_w', R') \in X_w(Y_m)$, $V_m(Y_m) = S(Y_m, Y_w, R) - V_w(Y_w, R) = S(Y_m, Y_w', R') - V_w(Y_w', R')$. Using the Envelope Theorem, $V_m'(Y_m) = S_1(Y_m, Y_w, R) = S_1(Y_m, Y_w', R')$.

V_w)) in Section 1.3.4. (V_m, V_w), $\Gamma_\sigma(V_m, V_w)$, and $\Gamma_\mu(\Gamma_\sigma(V_m, V_w))$ compose an equilibrium if $(V_m, V_w) \in \Gamma_V(\Gamma_\mu(\Gamma_\sigma(V_m, V_w)))$. Therefore, an equilibrium exists if the composite mapping $\Gamma_V \circ \Gamma_\mu \circ \Gamma_\sigma : (V_m, V_w) \twoheadrightarrow (V_m, V_w)$ has a fixed point. We can apply Glicksberg (1952) fixed-point theorem to show equilibrium. By Glicksberg, it suffices to show that the set of stable marriage payoff functions is nonempty, convex, and compact, the correspondence $\Gamma_V \circ \Gamma_\mu \circ \Gamma_\sigma$ is upper-hemicontinuous, and $\Gamma_V \circ \Gamma_\mu \circ \Gamma_\sigma(V_m, V_w)$ is nonempty, convex, and compact for each (V_m, V_w).

The primary difficulty is to show the compactness of the set of stable marriage payoff functions. The stability conditions and the Arzelà-Ascoli theorem imply compactness. By the Arzelà-Ascoli theorem, the set of stable marriage payoff functions is compact if and only if a sequence of stable marriage payoff functions is uniformly bounded and equicontinuous. Uniform boundedness follows directly from the boundedness of the surplus function. The equicontinuity relies on the stability condition. Take any stable marriage payoff functions V_m and V_w. Take a Y_m man who is matched to a (Y_w, R) woman. The stable marriage payoff functions satisfy $V_m(Y_m) = S(Y_m, Y_w, R) - V_w(Y_w, R)$. Any $Y'_m < Y_m$ man's marriage payoff $V_m(Y'_m) = \max_{(Y'_w, R') \in \text{supp}(\mu_w)}[S(Y'_m, Y'_w, R') - V_w(Y'_w, R')]$. The difference $V_m(Y_m) - V_m(Y'_m)$ is

$$[S(Y_m, Y_w, R) - V_w(Y_w, R)] - \max_{(Y'_w, R') \in \text{supp}(\mu_w)}[S(Y'_m, Y'_w, R') - V_w(Y'_w, R')]$$

which is smaller than

$$[S(Y_m, Y_w, R) - V_w(Y_w, R)] - [S(Y'_m, Y_w, R) - V_w(Y_w, R)] = S(Y_m, Y_w, R) - S(Y'_m, Y_w, R).$$

Since the difference $V_m(Y_m) - V_m(Y'_m)$ only depends on the surplus function S, it is equicontinuous.

The proof techniques can be adapted to any investments-and-matching model that uses the Becker-Shapley-Shubik matching model. Equilibrium can be shown to exist with these techniques in the models of Low (2015) and Chiappori et al. (2015b) that lack an equilibrium existence proof. Furthermore, note that the current proof only relies on the stability conditions of the marriage market, and does not rely on restrictions previous papers have hinged on, such as the dimensionality of the marriage characteristics, discreteness or continuity of the marriage characteristics support and supermodularity of the marriage surplus function.

Let me show two examples in which the existence proofs can be simplified by the current proof techniques. In Iyigun and Walsh (2007), men and women are endowed with asset. They can choose to divide up the endowment into current consumption and investment that enriches future consumption. Men and women are characterized by their investments in the marriage market and then match and bargain over the surplus exactly as in this setting. Equilibrium was shown to exist under an unnatural condition that cannot be easily verified. With our proof techniques, we can show that an equilibrium always exists. Construct the composite mapping from stable marriage payoff functions to the investment strategies to the distribution of marriage characteristics back to the space of marriage functions. We can apply Glicksberg's fixed-point theorem on the mapping to show existence, and the Arzelà-Ascoli theorem to take care of the compactness. In Chiappori et al. (2009), men and women born with heterogenous costs of education decide whether or not to go to college. Going to college delays entering the marriage market but makes a person an educated type. Agents are matched based on their education types and match-specific shocks. The authors explicitly characterize a quadruple of stable marriage payoffs of an educated/uneducated man/woman and the induced distributions of education and show equilibrium existence under different parameters. Instead, we can apply Glicksberg's fixed-point theorem as done here without the explicit characterization.

1.4 A Simple Model

I present a simplified version of the general framework in this section. Assume no discounting: $\delta = 1$. Let P_{i1} be degenerate: any agent without a college investment gets a low income Y_{iL}. Let $P_{i2}(\cdot|\theta_i) = P_{i3}(\cdot|\theta_i)$ with supports on Y_{iL} and Y_{iH}: after either investment, an agent gets a high-income offer with probability θ_i and a low-income offer with probability $1-\theta_i$. Equate investment costs too: $C_i = C_{i1} = C_{i2}$. Since an age 2 agent always accepts the high-income offer, agents' two decisions can be simply characterized by $\sigma_{i1}(\theta_i)$ the probability of making a college investment and $\sigma_{i2}(\theta_i)$ the probability of a career investment at age 2 conditional on receiving a low-income offer. Assume that it is a strictly dominant strategy for the highest ability agents to invest.[5] The women who enter the marriage market in the first period or the second period stay fit, but the women who enter the marriage market in the third period stay fit with probability ϕ_H and become less fit ($R_L < R_H$) with probability $\phi_L = 1-\phi_H$. Stationary and symmetric strategies σ_m and σ_w induce stationary measures μ_m on $\{Y_{mL}, Y_{mH}\}$ and μ_w on $\{Y_{wL}, Y_{wH}\} \times \{R_L, R_H\}$. Furthermore, assume that the surplus function is strictly supermodular in Y_m and Y_w and in Y_m and R.

Let us characterize the equilibrium. The optimal investments can be characterized by simple ability cutoffs. A θ_m-ability man makes a career investment if and only if

$$\underbrace{\theta_m[Y_{mH} - Y_{mL}]}_{\text{labor market gain}} + \underbrace{\theta_m[V_m(Y_{mH}) - V_m(Y_{mL})]}_{\text{marriage market gain}} - \underbrace{C_m}_{\text{investment cost}} \geq 0.$$

A man makes a career investment if his ability exceeds

$$\theta_{m2} = \frac{C_m}{Y_{mH} - Y_{mL} + V_m(Y_{mH}) - V_m(Y_{mL})}.$$

[5]The conditions that guarantee the strict dominance are $C_m < Y_{mH} - Y_{mL} + S(Y_{mH}, Y_{wL}, R_L) - S(Y_{mL}, Y_{wL}, R_L)$ and $C_w + \phi_L[S(Y_{mL}, Y_{wL}, R_H) - S(Y_{mL}, Y_{wH}, R_L)] < Y_{wH} - Y_{wL} + \phi_H[S(Y_{mL}, Y_{wH}, R_H) - S(Y_{mL}, Y_{wL}, R_H)]$.

A θ_m-ability man makes a college investment if and only if

$$\underbrace{\theta_m[Y_{mH} - Y_{mL}]}_{\text{labor market gain}} + \underbrace{\theta_m[V_m(Y_{mH}) - V_m(Y_{mL})]}_{\text{marriage market gain}} - \underbrace{C_m}_{\text{investment cost}}$$
$$+ \underbrace{(1-\theta_m)1_{\theta_m \geq \theta_{m2}}[\theta_m(Y_{mH} - Y_{mL} + V_m(Y_{mH}) - V_m(Y_{mL})) - C_m]}_{\text{net gain from a career investment}} \geq 0.$$

A θ_m-ability man makes a college investment if and only if

$$\theta_m[Y_{mH} - Y_{mL}] + \theta_m[V_m(Y_{mH}) - V_m(Y_{mL})] - C_m \geq 0.$$

Therefore, a man makes a college investment if his ability exceeds

$$\theta_{m1} = \frac{C_m}{Y_{mH} - Y_{mL} + V_m(Y_{mH}) - V_m(Y_{mL})},$$

the same as the career investment cutoff θ_{m2}. Let $\theta_m^* \equiv \theta_{m1} = \theta_{m2}$. In summary, ability $\theta_m \leq \theta_m^*$ men do not make any investment, and ability $\theta_m \geq \theta_m^*$ men make the college investment and make the career investment if they receive a low-income offer after the college investment.

Women's optimal investments can be similarly characterized. An ability θ_w woman makes a career investment if and only if

$$\underbrace{\theta_w[Y_{wH} - Y_{wL}]}_{\text{labor market gain}} + \underbrace{\theta_w[V_w(Y_{wH}, R_H) - V_w(Y_{wL}, R_H)]}_{\text{marriage market gain}} - \underbrace{C_w}_{\text{investment cost}}$$
$$- \underbrace{\phi_L[\theta_w(V_w(Y_{wH}, R_H) - V_w(Y_{wH}, R_L)) + (1-\theta_w)(V_w(Y_{wL}, R_H) - V_w(Y_{wL}, R_L))]}_{\text{fitness cost}} \geq 0.$$

When a woman makes a career investment, she pays not only an investment cost but also a fitness cost. With probability ϕ_L, she has lower fitness thus lower gain in the marriage market. A woman makes a career investment if and only if her ability exceeds

$$\theta_{w2} = \frac{C_w + \phi_L(V_w(Y_{wL}, R_H) - V_w(Y_{wL}, R_L))}{Y_{wH} - Y_{wL} + \phi_H V_w(Y_{wH}, R_H) + \phi_L V_w(Y_{wH}, R_L) - \phi_H V_w(Y_{wL}, R_H) - \phi_L V_w(Y_{wL}, R_L)}.$$

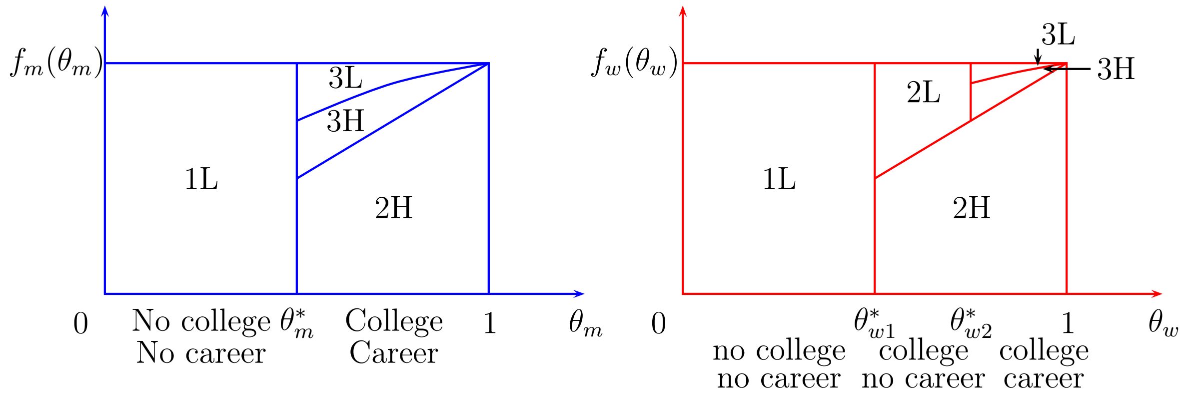

Figure 1.2: Men's and women's equilibrium strategy, marriage-age, and income.

A θ_w-ability woman's college investment yields an expected utility gain of

$$-C_w + \theta_w[Y_{wH} - Y_{wL}] + \theta_w[V_w(Y_{wH}, R_H) - V_w(Y_{wL}, R_H)] + (1-\theta_w)1_{\theta_w \geq \theta_{w2}} \times$$
$$\{-C_w + \theta_w[Y_{wH} - Y_{wL} + V_w(Y_{wH}, R_H) - V_w(Y_{wL}, R_H)] - \text{fitness cost}\}.$$

The marginal gain from a career investment is strictly smaller than the marginal gain from a college investment. Therefore, a woman makes a college investment if and only if her ability exceeds

$$\theta_{w1} = \frac{C_w}{Y_{wH} - Y_{wL} + V_w(Y_{wH}, R_H) - V_w(Y_{wL}, R_H)}.$$

Unlike men's equal thresholds, women's ability thresholds for the two investments are different. In summary, ability $\theta_w < \theta_{w1}$ women never invest, ability $\theta_{w1} \leq \theta_w < \theta_{w2}$ women make the college investment but never make the career investment, and ability $\theta_w \geq \theta_{w2}$ women make the college investment as well as the career investment.

The marriage market (μ_m, μ_w) induced by these optimal investments is summarized by six numbers: $\mu_m(Y_{mL})$, $\mu_m(Y_{mH})$, $\mu_w(Y_{wL}, R_L)$, $\mu_w(Y_{wH}, R_L)$, $\mu_w(Y_{wL}, R_H)$, $\mu_w(Y_{wH}, R_H)$. Low-income men in the marriage market consist of those who do not go to college and those who receive a low-income offer at age 3 after two unsuccessful investments,

$$\mu_m(Y_{mL}) = F_m(\theta_m^*) + \int_{\theta_m^*}^1 (1-\theta_m)^2 dF_m(\theta_m).$$

High-income men consist of those who go to college and receive a high-income offer at age 2 and those who reject a low-income offer after college and receive a high income-offer at age 3,

$$\mu_m(Y_{mH}) = \int_{\theta_m^*}^{1} [1 - (1 - \theta_m)^2] dF_m(\theta_m).$$

Fit low-income women include those who do not go to college, those who accept a low-income offer at age 2 after college and those who receive a low-income offer at age 3 after a career investment and stay fit,

$$\mu_w(Y_{wL}, R_H) = F_w(\theta_{w1}) + \int_{\theta_{w1}}^{\theta_{w2}} (1 - \theta_w) dF_w(\theta_w) + \phi_H \int_{\theta_{w2}}^{1} (1 - \theta_w)^2 dF_w(\theta_w).$$

Fit high-income women include those who receive a high-income offer at age 2, and a high-income offer at age 3 after a career investment and stay fit,

$$\mu_w(Y_{wH}, R_H) = \int_{\theta_{w1}}^{1} \theta_w dF_w(\theta_w) + \phi_H \int_{\theta_{w2}}^{1} (1 - \theta_w)\theta_w dF_w(\theta_w)$$

Less-fit high-income women are those who receive a high-income offer at age 3 after a career investment and become less-fit,

$$\mu_w(Y_{wH}, R_L) = \phi_L \int_{\theta_{w2}}^{1} [1 - (1 - \theta_w)^2] dF_w(\theta_w).$$

Less-fit low-income women are those who receive a low-income offer at age 3 after a career investment and become less-fit,

$$\mu_w(Y_{wL}, R_L) = \phi_L \int_{\theta_{w2}}^{1} (1 - \theta_w)^2 dF_w(\theta_w).$$

Now let me describe the stable matching. Since the surplus function is strictly supermodular in Y_m and Y_w and in Y_m and R, by Lemma 1, almost all (Y_{wH}, R_H) women marry higher-income husbands than any other women, and (Y_{wL}, R_L) women marry lower-income husbands than any other women. However, whether (Y_{wH}, R_L) or (Y_{wL}, R_H) women marry higher-income husbands depends on whether the following condition holds:

$$S(Y_{mH}, Y_{wL}, R_H) + S(Y_{mL}, Y_{wH}, R_L) > S(Y_{mH}, Y_{wH}, R_L) + S(Y_{mL}, Y_{wL}, R_H), \tag{1.2}$$

This is parallel to Condition 1.1 in the continuous case. Figure 1.3 illustrates the two possible scenarios. The left panel illustrates the case when (Y_{wL}, R_H) women marry higher-income husbands than (Y_{wH}, R_L) women, and the right panel illustrates the case when (Y_{wH}, R_L) women marry higher-income husbands than (Y_{wH}, R_L). Say $X_m(Y, R) \geq X_m(Y', R')$ if almost all (Y, R) women marry higher-income husbands than (Y', R') women.

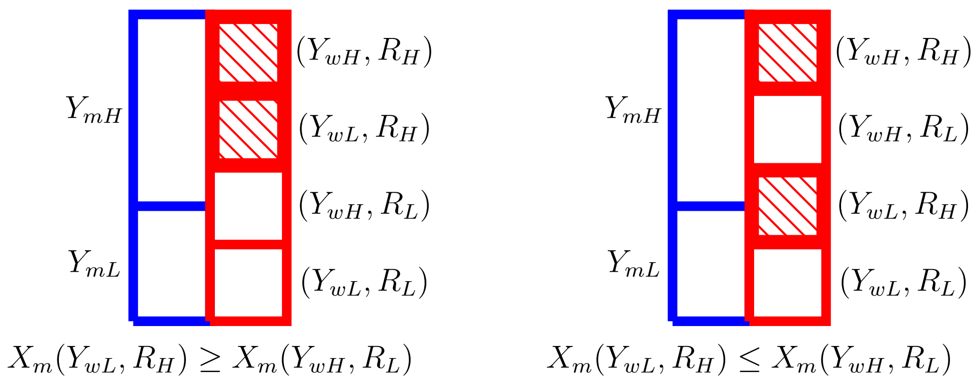

Figure 1.3: Stable matchings when the marriage characteristics take binary values.

Finally, the stability conditions pin down the marriage payoffs. For example, in the left panel of Figure 1.3, these conditions pin down the marriage payoff functions up to a constant,

$$V_m(Y_{mH}) - V_m(Y_{mL}) = S(Y_{mH}, Y_{wH}, R_L) - S(Y_{mL}, Y_{wH}, R_L),$$

$$V_w(Y_{wH}, R_H) - V_w(Y_{wL}, R_H) = S(Y_{mH}, Y_{wH}, R_H) - S(Y_{mH}, Y_{wL}, R_H),$$

$$V_w(Y_{wH}, R_H) - V_w(Y_{wH}, R_L) = S(Y_{mH}, Y_{wH}, R_H) - S(Y_{mH}, Y_{wH}, R_L),$$

$$V_w(Y_{wH}, R_L) - V_w(Y_{wL}, R_L) = S(Y_{mL}, Y_{wH}, R_L) - S(Y_{mL}, Y_{wL}, R_L).$$

Since agents base their investment decisions on marriage payoff difference between the high-income and the low-income, despite of indeterminacy in marriage payoffs, investments are unique.

Theorem 2. *An equilibrium* $(\sigma_m^*, \sigma_w^*, \mu_m^*, \mu_w^*, \mu^*, V_m^*, V_w^*)$ *exists uniquely, with the equilibrium marriage payoffs unique up to a constant.*

The proof explicitly shows that the equilibrium is unique under different parameters of the model. The proof idea is that an increase in equilibrium $\mu_m(Y_{mH})$ is accompanied by a decrease in equilibrium marriage payoffs difference $V_m(Y_{mH}) - V_m(Y_{mL})$. The difference between stable marriage payoffs $V_m(Y_{mH}) - V_m(Y_{mL})$ equals

$$\lambda[S(Y_{mH}, Y_{wL}, R_L) - S(Y_{mL}, Y_{wL}, R_L)] + (1-\lambda)[S(Y_{mH}, Y_{wH}, R_H) - S(Y_{mL}, Y_{wH}, R_H)] \equiv \Delta(\lambda)$$

for some $\lambda \in [0, 1]$. When $\lambda = 0$, the stable marriage payoffs difference $V_m(Y_{mH}) - V_m(Y_{mL}) = S(Y_{mH}, Y_{wH}, R_H) - S(Y_{mL}, Y_{wH}, R_H)$, which is only satisfied when $\mu_m(Y_{mH}) < \mu_w(Y_{wH}, R_H)$; when $\lambda = 1$, $V_m(Y_{mH}) - V_m(Y_{mL}) = S(Y_{mH}, Y_{wL}, R_L) - S(Y_{mL}, Y_{wL}, R_L)$, which is only satisfied when $\mu_m(Y_{mH}) > \mu_w(Y_{wH}, R_H) + \mu_w(Y_{wL}, R_H) + \mu_w(Y_{wH}, R_L)$. When $\mu_w(Y_{wH}, R_H) \leq \mu_m(Y_{mH}) \leq \mu_w(Y_{wH}, R_H) + \mu_w(Y_{wL}, R_H) + \mu_w(Y_{wH}, R_L)$, $V_m(Y_{mH}) - V_m(Y_{mL}) = \Delta(\lambda)$ for some $\lambda \in (0, 1)$, depending on the distributions of marriage characteristics. Each λ corresponds to the marriage market in which $V_m(Y_{mH}) - V_m(Y_{mL}) = \Delta(\lambda)$; there are fourteen possible marriage markets as enumerated in the proof.

Start with some λ. The stable marriage payoffs corresponding to λ induce optimal investments, and the optimal investments induce marriage characteristics distributions (μ'_m, μ'_w). The stable marriage payoffs difference $V'_m(Y_{mH}) - V'_m(Y_{mL})$ in the marriage market (μ'_m, μ'_w) equals $\Delta(\eta)$ for some $\eta \in [0, 1]$. η corresponds to the marriage market (μ'_m, μ'_w). For each λ, let $\eta(\lambda)$ denote the marriage market induced by the marriage market λ. λ^* is an equilibrium if $\lambda^* = \eta(\lambda^*)$. If $\eta(\lambda)$ is strictly decreasing in λ, then there is a unique solution λ^* to $\lambda = \eta(\lambda)$ and there is a unique equilibrium.

The reasoning for decreasing $\eta(\lambda)$ is as follows. When λ increases, $V_m(Y_{mH}) - V_m(Y_{mL})$ decreases, and $\mu_m(Y_{mH})$ increases: high-income men's relative value decreases if and only if high-income men become more abundant. When $V_m(Y_{mH}) - V_m(Y_{mL})$ decreases, men's incentive to make investments decreases and the mass of men making investments decreases. The mass of high-income men $\mu'_m(Y_{mH})$ in the induced marriage market $\eta(\lambda)$ decreases. $V'_m(Y_{mH}) - V'_m(Y_{mL}) = \Delta(\eta(\lambda))$ increases and $\eta(\lambda)$ decreases.

1.5 Implications

1.5.1 The College Gender Gap Puzzle

More and more women are going to college around the world, both in absolute numbers and in percentages. For example, in the United States, women's college enrollment rate has increased rapidly from 10% in 1970 to 45% in 2010. Several factors contribute to the rise. The labor market returns to college increased as the demand for skilled workers rose, the social norm for women to stay at home and take care of family has loosened as the home production technology (e.g., domestic time-saving appliances) has improved. The home production technology improvement has both freed women from the household to the work force as well as demanded more knowledge and education for everyday chores.

With so many factors contributing to the rise, it is not surprising to see that women's college enrollment rate has increased worldwide and has caught up with men's. As costs of investments, labor market opportunities, roles in the family converge for the two genders, we would expect women's college enrollment rate to converge to men's. However, it is very surprising to see that *more* women than men are going to college in most developed countries today. In 2014, women constituted 57% of college freshmen in the United States, and 58% in Canada. More women than men are in college in 30 out of 34 OECD countries.[6] The reversal of the college gender gap is indisputable by different measures such as entrance enrollment rate, graduation rates from two-year and four-year colleges.[7] It is not only surprising but

[6]Exceptions are Japan and Korea, and the two lowest-income countries Turkey and Mexico.

[7]See Goldin et al. (2006) for college education patterns of the United States over the last one hundred

also puzzling because women's labor market returns to college are not significantly higher than men's, and women on average still earn less than men in the workforce.

There have been many explanations for the reversal of the college gender gap. The most natural explanation is the gender difference in labor market returns to college. However, empirically the result is debatable (Dougherty, 2005; Mulligan and Rubinstein, 2008; Hubbard, 2011). If anything, men's labor market returns to college are higher than women's. Even if the gender difference in labor market returns can explain for example the United States' college gender gap, it is not likely that the labor markets across all developed countries are organized in similar ways. There should be other inherent gender differences that contribute to the reversal of the college gender gap. Goldin et al. (2006) and Becker et al. (2010a) argue that the gender difference is in non-cognitive abilities. Men's ability distribution is more spread than women's, but women have a higher average ability; therefore, when there are few college students, only the top students, consisted of mostly men, go to college, but when more people go to college, many more women of similar intermediate abilities go, causing the reversal of the college gender gap. Their evidence was the difference in high school grades. However, it is not always the case the college gender gap reverses at a certain enrollment rate. Moreover, high school grades can be interpreted as measures of investment efforts by parents instead of inherent abilities. There must be other factors that contribute to the reversal of the college gender gap.

I stress the gender difference in the marriage market returns to college. The marriage market returns to college have been found to be on the same order of magnitude as the labor market returns to college (Bruze, 2015). Chiappori et al. (2009) and Chiappori et al. (2015a) argue and present evidence for women's higher marriage market returns to college in the United States. The theoretical argument in Chiappori et al. (2009) can be further refined. In their model, more women than men go to college only when women are advantageous in

years. See Becker et al. (2010a) for college education patterns around the world since World War II.

some aspect: for example, women contribute more to the marriage surplus or earn more than men in the labor market. In contrast, in my model it is always the case that more women than men go to college even when men and women are the same except for that women face the fertility loss disadvantage that tampers with their career investment incentives.

Proposition 1. *Suppose that the setting is gender-symmetric except for the gender difference in fitness: men and women have the same ability distribution ($F_m = F_w = F$), have the same investment cost ($C_m = C_w = C$), face the same labor market opportunities ($P_{mH} = P_{wH} = P_H$ and $P_{mL} = P_{wL} = P_L$), contribute to the marriage surplus symmetrically ($S(Y_m, Y_w, R) = S(Y_w, Y_m, R)$), but women become less fit with positive probability ($\phi_L > 0$). In equilibrium, strictly more women than men go to college.*

Let me explain the logic of the result. When the setting is totally gender-symmetric ($\phi_H = 1$), every man and woman above a certain ability goes to college and makes a career investment after receiving a low-income offer. The same number of men and women go to college in equilibrium. When women face reproductive fitness loss ($\phi_H < 1$), men's and women's investment strategies differ. Whereas every college man still makes a career investment after receiving a low-income offer, not every college woman makes a career investment. As a result, fewer women than men end up with high-income jobs. But remember, the same people are also in the marriage market. The marriage payoffs are determined by supply and demand of different marriage characteristics. High-income women, compared to high-income men, are scarcer, and are therefore more valuable in the marriage market. The labor market gain from a college investment is the same for men and women, and the marriage market gain from a college investment is endogenously higher for women. A woman therefore gains more from a college investment.

This model makes the college gender gap puzzle less puzzling. We present an equilibrium model in which more women than men go to college but women still earn less than men on average. The gender difference in reproductive fitness and the marriage market play important roles in the result. Women's fitness cost in the third period directly discourages women from making career investments, but through the endogenous division of surplus in the marriage market the scarcity of high-income women indirectly encourages women to make college investments.

Let me elaborate. Figure 1.4 illustrates the returns to college for men and women of heterogeneous abilities. The highest-ability women has higher returns to college than the highest-ability men

$$Y_H - Y_L + V_w^*(Y_H, R_H) - V_w^*(Y_L, R_H) > Y_H - Y_L + V_m^*(Y_H) - V_m^*(Y_L).$$

The highest-ability women have higher marriage market returns and they do not worry about fitness constrain because they always succeed right out of college. All other intermediate-ability women need to worry about the fitness cost. Because of the fitness cost, some of the infra-marginal female college graduates with abilities between θ_{w1}^* and θ_{w2}^* have lower returns to college than the male college graduates with the same ability. However, the marginal θ_{w1}^*-ability female college graduates always receive higher total returns to college than men of the same ability. It crucial to distinguish infra-marginal and marginal college graduates' returns to college with a model of heterogeneous agents. The fitness cost constrains the infra-marginal female college graduates' career investment incentives but does not affect directly the marginal female college graduates' investment incentives. In fact, the fitness cost makes high-income women scarcer and through the endogenous determination of marriage payoffs indirectly raises the marginal women's college investment incentives.

The model with heterogeneous agents also calls for more care with estimating and interpreting average and marginal returns to college. Because some infra-marginal women have lower returns to college than the men with the same income-earning ability, the average marriage and labor market returns to college for women may be lower than for men even

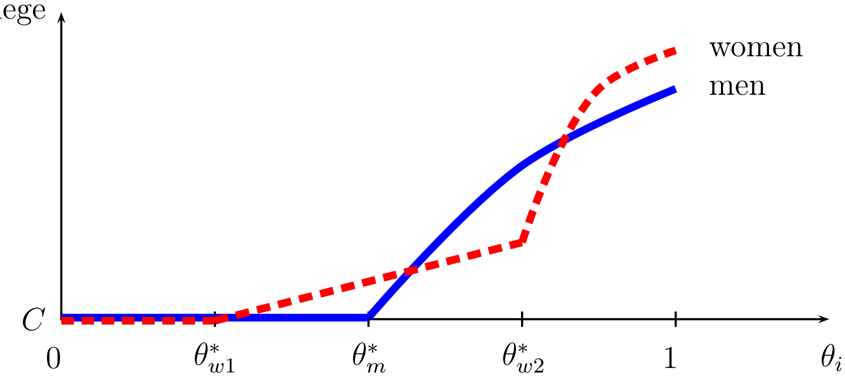

Figure 1.4: Equilibrium returns to college for men and women of heterogeneous abilities.

though the marginal returns to college are higher for women than for men, while more women than men go to college. In addition, because different populations of men and women enter college, the selection issue prevents us from simply estimating the average returns to college as the difference between the incomes of those who go to college and of those who do not.

Finally, let us the importance of strict surplus supermodularity in incomes ($\partial^2 S/\partial Y_m \partial Y_w > 0$) plays in Proposition 1. If the marriage surplus is modular in incomes ($\partial^2 S/\partial Y_m \partial Y_w = 0$), that is, there is no synergy between men and women in marriages, then always exactly the same number of men and women go to college. If the marriage surplus is strictly submodular in incomes ($\partial^2 S/\partial Y_m \partial Y_w < 0$), weakly more women than men go to college in equilibrium. The reversal of the college gender gap is unambiguous when the marriage surplus function is strictly supermodular in incomes. Strict supermodularity in incomes implies positive-assortative matching in incomes. The increase is empirically backed by the observed increase in positive-assortative matching in incomes as well as in education. The increase in positive-assortative matching is important to explain recent changes in marriage patterns and in income inequality (Stevenson and Wolfers, 2007; Greenwood et al., 2014). In the recent decades, as technological advances in home production free women from daily chores, the role of the household has gradually transitioned from a unit of labor specialization (a submodular surplus function) to a unit of labor cooperation (a supermodular surplus func-

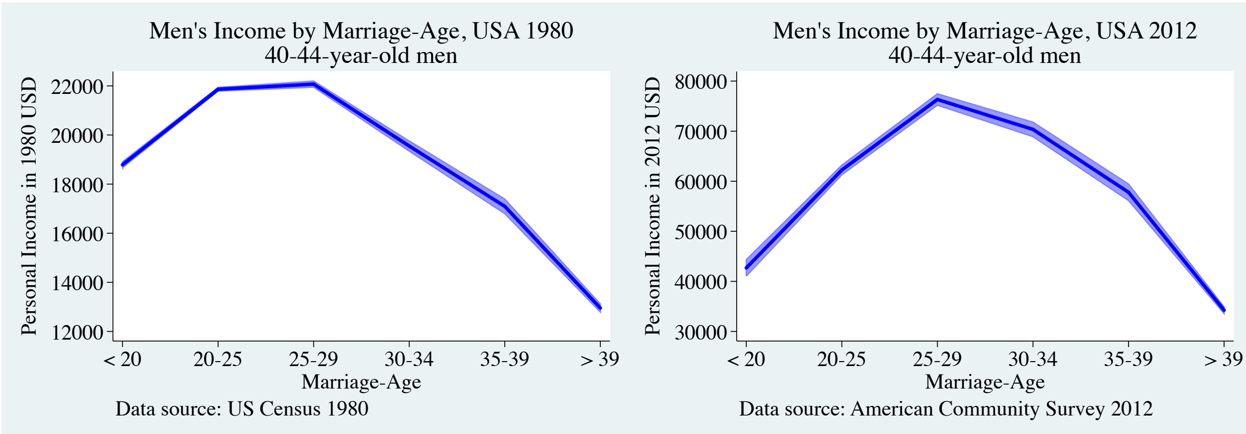

Figure 1.5: The relationships between marriage-age and income among American men ages 40-44 in 1980 and in 2012.

tion). When love and companionship are valued more, a husband and a wife of similar education and income levels have more to share in their life and leisure, so the consideration of complementing each other in intellectual activities is stronger than the consideration of substituting each other for housework for example. In addition, human capital investments in children have also become more important. The benefits multiply when both parents are highly educated. Therefore, the transition of the role of marriage and family from a unit of labor specialization to a unit of labor cooperation also serves as a potentially crucial reason for the reversal of the college gender gap.

1.5.2 A Man's Marriage-Age and His Income

Across time and countries, the relationship between a man's age at first marriage and his income later in life has been hump-shaped: those who married between 25 and 30 earned more on average than those who married earlier and later. In addition, those who remain unmarried into their forties earned the least. Figure 1.5 shows the relationship among American men ages 40-44 in 1980 and in 2012. Figure 1.10 shows the same relationship among Canadian men ages 40-44 in 1981 and among Brazilian men ages 40-44 in 1991.

Previous studies on the relationships between marriage-age and income cannot explain the non-monotonic relationships. Furthermore, they are based on the outdated premise of Becker (1973) that marriage and household formation allow men to specialize in labor-market activities and women in household chores. Keeley (1974, 1977, 1979), using a search-theoretic framework, predicts a negative marriage-age/income relationship because higher-income men benefit more from marriage specialization and find partners more easily. Bergstrom and Bagnoli (1993), using a signaling model, predict a positive relationship for men. They argue that it takes time for men to reveal their true income-earning abilities, so men with higher income-earning potential delay marriage.

The current model always predicts a hump-shaped relationship.

Proposition 2. *The relationship between a man's marriage-age and his income in equilibrium is always hump-shaped: men who marry in the second period have higher average income than those who marry in the first period and those who marry in the third period.*

Figure 1.2 illustrates men's equilibrium strategies, marriage-age and incomes. Those marrying in the first period are the low-ability men who do not go to college and receive a low income. Those marrying in the second period are the high-ability men who succeed with a high income after a college investment. Those marrying in the third period are the high-ability men who receive a low-income offer after college and make a career investment to try to improve. Some of them succeed and receive a high income, while some of them fail and receive a low income. On average, those marrying in the third period earn more than those marrying in the first period but less than those marrying in the second period.

The upward-sloping portion of the hump-shaped curve comes from the fact that college investment is time-intensive and delays marriage. Those men who marry early do not have a bright labor market outlook and choose to marry as early as possible. Those who can have a brighter labor market outlook put in effort and take time to improve their chance of success and delay marriage. The positive relationship is the same as in Bergstrom and Bagnoli (1993) but the driving force is conceptually very different. Whereas in Bergstrom and Bagnoli (1993), men delay marriage to reveal their private information of income-earning abilities, in this model, everything is public information, and people delay marriage to improve their income.

The downward-sloping portion of the hump-shaped curve among men who marry relatively late formalizes the effect mentioned in the concluding remarks of Bergstrom and Schoeni (1996), the first paper documenting men's hump-shaped relationship between marriage-age and income, *"Some of these men who marry very late in life ... may be persons whose successes in life have not met the expectations that led them to postpone marriage and who continue to postpone marriage until their true worth is recognized."* This marriage-delaying effect is also conceptually different from the marriage-market friction effect of Becker-Keeley. The marriage delay in this model is due to the interaction of investments and the marriage market. A less desirable outcome in the labor market incentivizes men to marry later but a desirable outcome in the labor market incentivizes men to marry early.

My model also predicts that the lowest-income men will stay unmarried if there is an unequal number of men and women in the marriage market. This prediction is also consistent with the evidence presented by Figures 1.5 and 1.10. It formalizes another remark in Bergstrom and Schoeni (1996): *"There may also be a considerable number of men who are such poor marriage material, that any woman whom they would wish to marry would prefer being single to marrying one of these men."*

1.5.3 A Woman's Marriage-Age and Her Income

Figure 1.6 shows the relationships between marriage-age and income among American

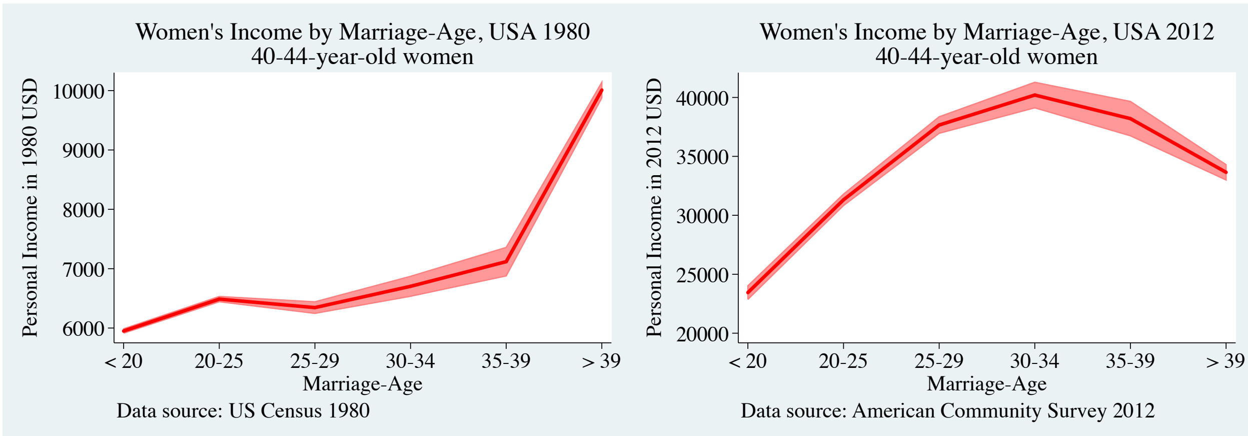

Figure 1.6: The relationship between marriage-age and income among American women ages 40-44 in 1980 and in 2012.

women ages 40-44 in 1980 and in 2012. The relationship was positive in 1980: the later a woman married, the more she earned. The same positive relationship was observed in Canada and Brazil in 1981 and 1991 respectively (Figure 1.11). However, in 2012, women's relationship between marriage-age and income became similarly hump-shaped to men's: those who married after 35 earned less than those married before 35. The same relationships are unaltered if we restrict the sample to different races or to only those who earned a positive income.

Previous models could not predict the non-monotonic relationship and the change over time. Becker (1973) and Keeley (1974, 1977, 1979), with their marriage-market friction effect, predict that because low-income women benefit more from household specialization, they marry earlier, and the marriage-age/income relationship is positive. Bergstrom and Bagnoli (1993) on the other hand predict no relationship between women's income and marriage-age, because every woman marries in the first period in their two-period model. The current model predicts the change from the positive relationship to the non-monotonic relationship.

Proposition 3. *When the fitness cost is sufficiently large (ϕ_H is sufficiently small and/or*

Condition 1.2 holds), the equilibrium marriage-age/income relationship for women is positive: the average income of women increases with their marriage-age. When the fitness cost is sufficient small (ϕ_H is sufficiently large and/or Condition 1.2 does not hold), the equilibrium relationship is hump-shaped: the women who marry in the second period have higher average income than those who marry in the first period and those who marry in the third period.

Figure 1.2 shows women's equilibrium strategies, marriage-age, and income when their income abilities are uniformly distributed. Low-ability women marry in the first period and all earn a low income. Those who marry in the second period consist of all the intermediate-ability (between θ^*_{w1} and θ^*_{w2}) women who do not make a career investment regardless of the outcome of the college investment, and the high-ability women who receive a high income after the college investment. Women who marry in the third period are those of high-ability who receive a low-income offer after college and make a career investment.

The women who marry in the second period may have higher or lower average income than the women who marry in the third period. When the fitness cost is high, only extremely high-ability unlucky women make the career investment and delay marriage to the third period. Many of those who marry in the second period are low-income earners. When the fitness cost is low, more women make career investments but many of them marry with a low income in the third period. In the extreme case when men and women do not have any difference in reproductive fitness, women's equilibrium strategies are the same as men's, and women's marriage-age/income relationship becomes as hump-shaped as men's. Among the women who marry in the second period, not many will earn a low income. Among the women who marry in the third period, many are of intermediate-ability rather than of extremely-high ability, and they will earn a low income. The average income of those who marry in the third period can be lower than the average income of those who marry in the second period, and the marriage-age/income relationship is hump-shaped.

Several factors in the model can contribute to the change in relationships from positive in 1980 to hump-shaped in 2012. The declining fitness cost in the marriage market results from the interplay between social and economic changes that push up the supply of and down the demand for women's reproductive fitness. On one hand, the supply of reproductive fitness has increased. Advances in medical technology (e.g. in-vitro fertilization, egg-freezing) and better maternal health services result in higher probability of staying fit, so older women can have children with more ease and less adverse health effects. On the other hand, the demand for reproductive fitness has decreased. For instance, the desired family size has decreased. In the United States, the average desired number of children has declined from 3.6 to 2.6 from 1960 to 2010. Demand for women's reproductive fitness decreases because increases in parents' incomes as well as higher opportunity cost of raising kids prevent bigger family size. Many families have shifted from demand for quantity of children to demand for quality (Becker and Lewis, 1973). Women's reproductive fitness becomes less of a concern than women's income and education. Finally, a simple increase in income gain from college and career investments also has the same effect as a decrease in the relative importance of reproductive fitness: when income gain becomes much more important, gender difference in the marriage market becomes less important.

1.5.4 A Woman's Marriage-Age and Her Husband's Income

Figure 1.7 shows the relationship between a woman's marriage-age and her husband's income. It was persistently hump-shaped: women who married around the median marriage-age (22 in 1980, 26 in 2012) married husbands with the highest income on average. One-dimensional assortative matching by incomes could not explain the non-monotonic relationship. In 1980, as shown in Figure 1.6, the later a woman married, the more she earned on average. If men and women are simply positive-assortatively matched by incomes, the later

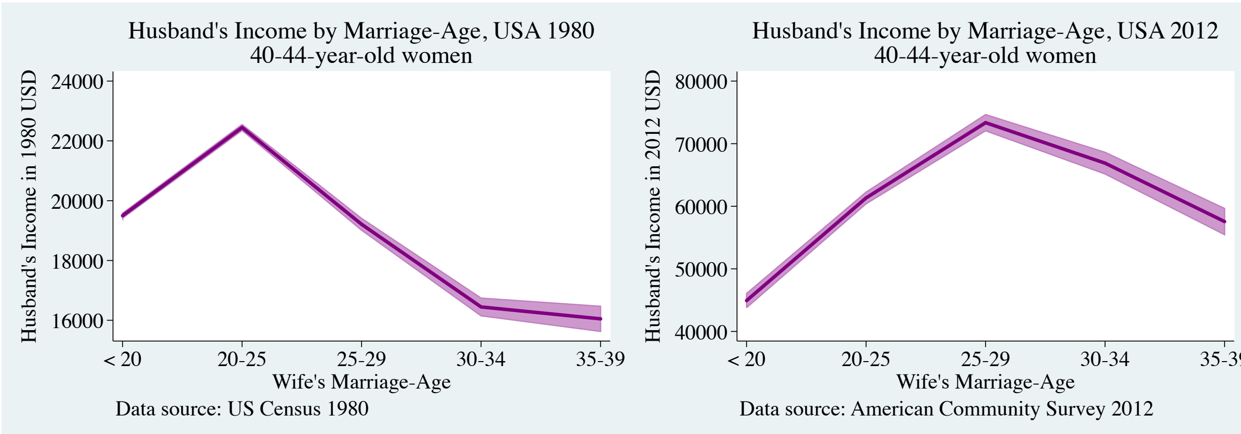

Figure 1.7: The relationship between a woman's marriage-age and her husband's income among American women ages 40-44 was hump-shaped in both 1980 and 2012.

a woman marries, the more her husband would earn. If men and women are simply negative-assortatively matched by incomes, the later a woman marries, the less her husband would earn. Furthermore, such an assortative matching model cannot explain another change. In 1980, women who married before the median marriage-age had higher-income husbands than those who married after the median marriage-age, but in 2012, women who married after the median marriage-age had higher-income husbands than those who married before the median marriage-age.

The extra dimension of reproductive fitness in the marriage market helps explain the non-assortative matching as well as the change over time.

Proposition 4. *The relationship between a woman's marriage-age and her husband's income in equilibrium is always hump-shaped. When the fitness cost is sufficiently high (ϕ_H is sufficiently small and/or Condition 1.2 holds), women who marry in the first period have higher-income husbands on average than women who marry in the third period. When the fitness cost is low, (ϕ_H is sufficiently large and/or Condition 1.2 does not hold), the women who marry in the first period have lower-income husbands on average than the women who marry in the third period.*

The women who marry in the first period have low average income but are reproductively fit. The women who marry in the second period are reproductively fit and college-educated; a significant portion earns a high income. The women who marry in the second period earn a higher average income than those who marry in the first period, and all of them are reproductively fit. By Lemma 1, because higher-income fit women marry higher-income men, the women who marry in the second period have higher-income husbands on average than the ones who marry in the first period. The women who marry in the third period have relatively higher income but lower reproductive fitness on average than those who marry in the first period.

The relative importance of reproductive fitness in the marriage market determines whether those who marry in the first period or in the third period have higher-income husbands on average. When the relative importance of reproductive fitness is so high that in the equilibrium matching fit women marry higher-income husbands than almost all less-fit women, then those who marry in the third period marry lower-income husbands than those who marry in the first period. Otherwise, if the relative importance of reproductive fitness is not as high when Condition 1.2 does not hold, it is possible that higher income can compensate for lower fitness in the marriage market, so that higher-income fit women may marry higher-income husbands than low-income less-fit women. In addition, when ϕ_H is sufficiently large, the average income of the husbands of the women who marry at age 3 is larger than marriage-age 1 women's husbands' average income.

The right-skewed equilibrium relationship when reproductive fitness is important in the marriage market is consistent with the observed relationship in 1980. The left-skewed equilibrium relationship when reproductive fitness is less important in the marriage market is consistent with the observed relationship in 2012. In 1980, higher-income men in general married low-income fit women but recently higher-income men marry high-income fit women. The factors that have contributed to the change from women's positive marriage-age/income

Figure 1.8: Predicted relationships between a man's marriage-age and his income, between a woman's marriage-age and her income, and between a woman's marriage-age and her husband's income.

relationship in 1980 to the hump-shaped relationship in 2012 also contribute to the change from the right-skewed equilibrium relationship to the recent left-skewed relationship: the supply of reproductive fitness increased and the demand decreased.

We can calculate the equilibrium relationships between a man's marriage-age and his personal income, between a woman's marriage-age and her personal income, and between a woman's marriage-age and her husband's income. Figure 1.8 graphs the equilibrium relationships under two different regimes: $\phi_H = 0.6$ in the first setting and $\phi_H = 0.8$ in the second setting, while all the other parameters remain constant ($F_m(\theta) = F_w(\theta) = \theta$, $C_m = C_w = 1$, $Y_{mH} = 3$, $Y_{mL} = 2$, $Y_{wH} = 2.4$, $Y_{wL} = 1.6$, $S(Y_m, Y_w, R) = 1_{R=R_H}(Y_m Y_w/2 - 1)$). The equilibrium relationships are consistent with the observed relationships in 1980 and 2012, respectively.

1.5.5 Income Uncertainty Delays Marriage

So far in the model, making an investment and entering the marriage market in the same period is not allowed. In this section, I relax this restriction and allow men and women to enter the marriage market and invest at the same time. The income uncertainty associated with the investment are not resolved when people enter the marriage market. Their marriage characteristic cannot be represented by a real number of realized income. The marriage characteristic may be an income distribution from which the realized income is drawn. The basic theoretical framework needs to be expanded in order to include not only new strategies but also new marriage characteristics.

The result is quite surprising:

Proposition 5. *When an agent is sufficiently patient, investing and delaying marriage strictly dominates investing and marrying at the same time.*

Intuitively, entering the marriage market early without realizing his marriage characteristics locks a man to a wife early. If he waits to marry, he can marry a wife who is more suitable to him and who can provide him higher marriage payoff than the wife he is locked to early. The same logic applies to a woman. Remember that in any stable matching, an agent is always matched with the partner that gives him or her the higher possible marriage payoff. The wife he marries after he realizes his income is always more suitable in this sense. Therefore, when the flow marriage payoff is negligible, a man always has the incentive to delay marriage and wait for the income uncertainty to resolve to marry a more suitable wife.

This income-uncertainty effect is conceptually different from the Bergstrom-Bagnoli signaling effect although both effects delay marriage. In Bergstrom and Bagnoli (1993), a man delays marriage to let his privately known income-earning ability become public information in the second period, and only high-ability men have incentives to delay marriage. In this model, although all the information is public, income uncertainty coupled with the competitive structure of the marriage market deters all people from marrying early. The result highlights a marriage-delaying effect caused by the competitive nature of the marriage market. Oppenheimer (1988) argues that people tend to marry only after income uncertainty is resolved, and my model formalizes the argument by showing the undesirability of marrying early.

It offers an explanation to why overwhelming majority of people only marry after they have finished their schooling. Among people ages 11-21 in 1979 NLSY, 79% of women and 84% of men married after they have finished all schooling (Browning et al., 2014). Nonetheless, not everyone waits until he or she finishes all the schooling as predicted by the model. Possible reasons for early marriage suggested by the current model include that agents enjoy the flow consumption of marriage and their uncertainty about future income is quite low.

1.6 Conclusion

This paper provides a dynamic equilibrium framework to study simultaneously human capital investments and the marriage market. College and career investment decisions, distributions of marriage characteristics, matching between men and women, and the division of surplus within a household are all endogenized. I provide equilibrium existence proof techniques applicable to other investment-and-matching models in the literature and also an equilibrium uniqueness theorem in a specialized model. I am able to derive a set of implications about the labor market and the marriage market. First, the model explains the college gender gap puzzle that more women go to college but earn less by arguing that women have an endogenously higher marriage market return to college. Second, the model is consistent with men's hump-shaped relationship between marriage-age and income, and the shift in women's relationship from positive to hump-shaped. Third, the stable matching is non-assortative in incomes, and explains how higher-income husbands become increasingly inclined to marry higher-income women than younger lower-income women. Finally, I extend the basic model with post-marital investments, and show that uncertainty about future income can deter people from marrying early.

The current framework serves as a foundation for future theoretical, applied, and empirical work. First, we can enrich the current framework by incorporating cohabitation, divorce, and household consumption. Second, the model also makes predictions about different effects on marriage-age and the marriage matching by ages of men and women. Other implications should be explored. Finally, we can try to estimate the investment-and-matching models with techniques developed by Choo and Siow (2006).

1.7 Proofs

1.7.1 Proofs of Lemmas 1 and 2

Proof of Lemma 1. The proof is by way of contradiction. First we look at the case when the marriage surplus function is strictly supermodular in Y_m and Y_w. Suppose that there is a positive measure of couples of (Y_m, Y'_w, R) and (Y'_m, Y_w, R) where $Y'_m > Y_m$ and $Y'_w > Y_w$. The marriage surplus generated by the two matched couples is $S(Y_m, Y'_w, R) + S(Y'_m, Y_w, R)$. The surplus generated by swapping the partners is $S(Y_m, Y_w, R) + S(Y'_m, Y'_w, R)$. The surplus generated by the swap is strictly higher because the surplus function is strictly supermodular in incomes. Since there is a positive measure of such couples, the total surplus cannot be maximized under the stable matching, contradicting the surplus-maximizing property of the stable matching. We can analogously prove the negative-assortative matching when the marriage surplus function is strictly submodular in Y_m and Y_w. $\square$

Proof of Lemma 2. First, I show that Condition 1.1 is necessary for the pure matching to be stable. Suppose that Condition 1.1 does not hold. Then there exist Y_m and $\epsilon > 0$ such that for almost all $Y' \in (Y_m - \epsilon, Y_m + \epsilon)$, $Y \in (Y_m^* - \epsilon, Y_m^*)$, $Y_w \in (Y_{wH} - \epsilon, Y_{wH})$,

$$S(Y', G_{wH}^{-1}(G_m(Y) - G_{wL}(Y_w)), R_H) + S(Y, Y_w, R_L)$$

$$< \quad S(Y, G_{wH}^{-1}(G_m(Y) - G_{wL}(Y_w)), R_H) + S(Y', Y_w, R_L).$$

It is more efficient to have couples $(Y', (G_{wH}^{-1}(G_m(Y) - G_{wL}(Y_w)), R_H))$ and (Y, Y_w, R_L) than couples $(Y, (G_{wH}^{-1}(G_m(Y) - G_{wL}(Y_w)), R_H))$ and (Y', Y_w, R_L). Hence the pure matching with couples $(Y, (G_{wH}^{-1}(G_m(Y) - G_{wL}(Y_w)), R_H))$ and (Y', Y_w, R_L) is not efficient and therefore not stable.

Next, I prove sufficiency of Condition 1.1 for the pure matching to be stable. When Condition 1.1 holds, it is never more efficient for (Y_{wH}, R_H) to marry any income $Y_m > Y_m^*$ men than to marry income Y_m^* men. The highest-income husband a (Y_{wH}, R_L) can marry is Y_m^*. By Lemma 1, for almost all Y_w, a (Y_w, R_H) woman marries a higher-income husband than a (Y_w, R_L) woman, and a (Y_{wH}, R_L) woman marries a higher income husband than a (Y_w, R_L) woman with any income $Y_w < Y_{wH}$. Therefore, the lowest-income husband a (Y_{wH}, R_L) woman can marry is also Y_m.

Finally, I show that thresholds Y_m^+ and Y_m^- exist. Because $S(Y_m, Y_w, R)$ is strictly supermodular in Y_w and R, $S_1(Y_{mH}, Y_{wH}, R_H) > S_1(Y_{mH}, Y_{wH}, R_L)$. By the smoothness of the surplus function, for some $\epsilon > 0$, for almost all $Y > Y_{mH} - \epsilon$ and $Y_w > Y_w - \epsilon$, $S_1(Y, Y_w, R_H) > S_1(Y, Y_{wH}, R_L)$. That is, for any $Y_m < Y_m' < Y_{mH} - \epsilon$,

$$\int_{Y_m}^{Y_m'} S_1(\widetilde{Y}_m, Y_w, R_H) d\widetilde{Y}_m > \int_{Y_m}^{Y_m'} S_1(\widetilde{Y}_m, Y_{wH}, R_L) d\widetilde{Y}_m,$$

or equivalently for any Y_m and $Y_m' > Y_m$,

$$S(Y_m', Y_w, R_H) - S(Y_m, Y_w, R_H) > S(Y_m', Y_{wH}, R_L) - S(Y_m, Y_{wH}, R_L).$$

Therefore, the marriage surplus generated by couples (Y_m', Y_w, R_H) and (Y_m, Y_{wH}, R_L) is strictly more than the marriage surplus generated by couples (Y_m, Y_w, R_H) and (Y_m', Y_{wH}, R_L). The existence of Y_m^- can be similarly shown from $S_1(Y_{mL}, Y_{wL}, R_H) > S_1(Y_{mL}, Y_{wL}, R_L)$. □

1.7.2 Proof of Theorem 1

By Glicksberg, it suffices to show that the space of stable marriage payoff functions (V_m, V_w) is non-empty, compact, and convex, $\Gamma_V \circ \Gamma_\mu \circ \Gamma_\sigma$ is upper-hemicontinuous, and $\Gamma_V(\Gamma_\mu(\Gamma_\sigma(V_m, V_w)))$ is non-empty, compact, and convex for any pair of stable marriage payoff functions (V_m, V_w).

First, I show that the space of stable marriage payoff functions is non-empty, compact, and convex. Non-emptiness follows from the existence of a stable outcome for every marriage market (Gretsky et al., 1992). To show compactness of the space of stable marriage payoff functions (V_m, V_w), we need to show that if $(\mu_m^n, \mu_w^n) \to (\mu_m, \mu_w)$, then the sequence $(V_m^n, V_w^n)_n$ where $(V_m^n, V_w^n) \in \Gamma_V(\mu_m^n, \mu_w^n)$ for each n has a convergent subsequence. To show that $(V_m^n, V_w^n)_n$ has a convergent subsequence, by the Arzelà-Ascoli Theorem, it suffices to show that (V_m^n, V_w^n) is a uniformly bounded, equicontinuous sequence of real-valued functions. It is uniformly bounded because the surplus function is increasing in every argument and $V_m^n(Y_m)$, $V_w^n(Y_w, R) \leq S(Y_{mH}, Y_{wH}, R_H)$. Equicontinuity follows from the following crucial argument. For each Y_m and n, there exists $(Y_w^n, R^n) \in \text{supp}(\mu_w^n)$ such that

$$V_m^n(Y_m) = S(Y_m, Y_w^n, R^n) - V_w^n(Y_w^n, R^n).$$

For any $Y_m > Y_m'$, because

$$V_m^n(Y_m') = \max_{(Y_w, R) \in \text{supp}(\mu_w^n)} [S(Y_m', Y_w, R) - V_w(Y_w, R)],$$

we have

$$\begin{aligned}
& V_m^n(Y_m) - V_m^n(Y_m') \\
= \;& S(Y_m, Y_w^n, R^n) - V_w^n(Y_w^n, R^n) - \max_{(Y_w, R) \in \text{supp}(\mu_w^n)} [S(Y_m', Y_w, R) - V_w(Y_w, R)] \\
\leq \;& [S(Y_m, Y_w^n, R^n) - V_w^n(Y_w^n, R^n)] - [S(Y_m', Y_w^n, R^n) - V_w^n(Y_w^n, R^n)]
\end{aligned}$$

$$
\begin{aligned}
&= S(Y_m, Y_w^n, R^n) - S(Y'_m, Y_w^n, R^n) \\
&= \frac{S(Y_m, Y_w^n, R^n) - S(Y'_m, Y_w^n, R^n)}{Y_m - Y'_m}(Y_m - Y'_m).
\end{aligned}
$$

$[S(Y_m, Y_w^n, R^n) - S(Y'_m, Y_w^n, R^n)]/(Y_m - Y'_m)$ is bounded by some real number K because S is continuous on a bounded set. $V_w^n(\cdot)$ is a uniformly bounded and equicontinuous sequence of real-valued functions by the same argument.

The space of stable marriage payoff functions (V_m, V_w) is convex if and only if for any $(V_m^1, V_w^1) \in \Gamma_V(\mu_m^1, \mu_w^1)$, $(V_m^2, V_w^2) \in \Gamma_V(\mu_m^2, \mu_w^2)$, and $\lambda \in (0,1)$, there exists (μ_m, μ_w) such that $(\lambda V_m^1 + (1-\lambda)V_m^2, \lambda V_w^1 + (1-\lambda)V_w^2) \in \Gamma_V(\mu_m, \mu_w)$. Let $\Sigma(\mu_m, \mu_w)$ represent the value of the primal linear programming problem PP(μ_m, μ_w) and also the value of the dual linear programming problem DP(μ_m, μ_w) since the values of the two linear programs coincide (Gretsky et al., 1992). Because Σ is weakly continuous, there exists a (μ_m, μ_w) such that $\text{supp}(\mu_m) \subset \text{supp}(\mu_m^1) \cup \text{supp}(\mu_m^2)$, $\text{supp}(\mu_w) \subset \text{supp}(\mu_w^1) \cup \text{supp}(\mu_w^2)$, and it satisfies $\Sigma(\mu_m, \mu_w) = \lambda\Sigma(\mu_m^1, \mu_w^1) + (1-\lambda)\Sigma(\mu_m^2, \mu_w^2)$. For any (Y_m, Y_w, R), $(\lambda V_m^1 + (1-\lambda)V_m^2)(Y_m) \geq 0$, $(\lambda V_w^1 + (1-\lambda)V_w^2)(Y_w, R) \geq 0$, and for all $Y_m \in \text{supp}(\mu_m)$ and $(Y_w, R) \in \text{supp}(\mu_w)$, $(\lambda V_m^1 + (1-\lambda)V_m^2)(Y_m) + (\lambda V_w^1 + (1-\lambda)V_w^2)(Y_w, R) \geq S(Y_m, Y_w, R)$. Therefore, $(\lambda V_m^1 + (1-\lambda)V_m^2, \lambda V_w^1 + (1-\lambda)V_w^2)$ solves the dual linear programming problem DP(μ_m, μ_w), and is thus in the space of stable marriage payoff functions.

Second, I show that the composite mapping $\Gamma_V \circ \Gamma_\mu \circ \Gamma_\sigma$ is upper-hemicontinuous by showing that $\Gamma_\mu \circ \Gamma_\sigma$ is a continuous function, and Γ_V is an upper-hemicontinuous correspondence. Continuity of $\Gamma_\mu \circ \Gamma_\sigma$ follows from the characterization of Γ_σ by ability cutoffs and reservation income, and the continuity of the ability cutoffs and the reservation income in marriage payoff functions (V_m, V_w). Since the sequence $(V_m^n, V_w^n)_n$ has a convergent subsequence, Γ_V is upper-hemicontinuous if and only if $(\mu_m^n, \mu_w^n) \to (\mu_n, \mu_w)$, $(V_m^n, V_w^n) \in \Gamma_V(\mu_m^n, \mu_w^n)$, and $(V_m^n, V_w^n) \to (V_m, V_w)$ implies $(V_m, V_w) \in \Gamma_V(\mu_m, \mu_w)$. It suffices to show that $\Sigma(\mu_m, \mu_w) = \int V_m d\mu_m + \int V_w d\mu_w$, and (V_m, V_w) satisfy the conditions that $V_m(Y_m) \geq 0$,

$V_w(Y_w, R) \geq 0, V_m(Y_m) + V_w(Y_w, R) \geq S(Y_m, Y_w, R)$ for all (Y_m, Y_w, R).

$$\begin{aligned}
& |\int V_m d\mu_m + \int V_w d\mu_w - \Sigma(\mu_m, \mu_w)| \\
= \; & |\int V_m d\mu_m + \int V_w d\mu_w - \int V_m^n d\mu_m^n - \int V_w^n d\mu_w^n + \Sigma(\mu_m^n, \mu_w^n) - \Sigma(\mu_m^n, \mu_w^n)| \\
\leq \; & |\int V_m d\mu_m + \int V_w d\mu_w - \int V_m^n d\mu_m^n - \int V_w^n d\mu_w^n| + |\Sigma(\mu_m^n, \mu_w^n) - \Sigma(\mu_m, \mu_w)| \\
\leq \; & |\int V_m d\mu_m + \int V_w d\mu_w - \int V_m^n d\mu_m^n - \int V_w d\mu_w^n| + |\Sigma(\mu_m^n, \mu_w^n) - \Sigma(\mu_m, \mu_w)| \\
& +|\int V_m^n d\mu_m^n + \int V_w^n d\mu_w^n - \int V_m^n d\mu_m^n - \int V_w^n d\mu_w^n| \\
\leq \; & |\int V_m d\mu_m + \int V_w d\mu_w - \int V_m d\mu_m^n - \int V_w d\mu_w^n| \\
& +\|V_m - V_m^n\| + \|V_w - V_w^n\| + |\Sigma(\mu_m^n, \mu_w^n) - \Sigma(\mu_m, \mu_w)|.
\end{aligned}$$

Fix any ϵ. Since $(\mu_m^n, \mu_w^n) \to (\mu_m, \mu_w)$, there is an $N_1(\epsilon)$ such that for all $n > N_1(\epsilon)$, $|\int V_m d\mu_m + \int V_w d\mu_w - \int V_m d\mu_m^n - \int V_w d\mu_w^n| \leq \epsilon/3$. Since $(V_m^n, V_w^n) \to (V_m, V_w)$, there is an $N_2(\epsilon)$ such that for all $n > N_2(\epsilon)$, $\|V_m - V_m^n\| + \|V_w - V_w^n\| \leq \epsilon/6$. Since Σ is weakly continuous, there is $N_3(\epsilon)$ such that for all $n > N_3(\epsilon)$, $|\Sigma(\mu_m^n, \mu_w^n) - \Sigma(\mu_m, \mu_w)| \leq \epsilon/3$. Therefore, given any ϵ, for all $n > N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon), N_3(\epsilon)\}$, $|\int V_m d\mu_m + \int V_w d\mu_w - \Sigma(\mu_m, \mu_w)| \leq \epsilon$. Therefore, $|\int V_m d\mu_m + \int V_w d\mu_w - \Sigma(\mu_m, \mu_w)| = 0$. We need to show that it also satisfies the stability conditions. Suppose otherwise: $V_m(Y_m) + V_w(Y_w, r) < S(Y_m, Y_w, R)$ for a positive measure of (Y_m, Y_w, R). Then $\lim_{n\to\infty}[\|V_m - V_m^n\| + \|V_w - V_w^n\|] > 0$, a contradiction to the assumption that $V_m^n \to V_m$ and $V_w^n \to V_w$.

Finally, I show that $\Gamma_V(\Gamma_\mu(\Gamma_\sigma(V_m, V_w)))$ is non-empty, compact, and convex. Non-emptiness of Γ_μ and Γ_σ follows from constructions, and non-emptiness of Γ_V is proven by Gretsky et al. (1992). Remember that stable marriage payoff functions are solutions to the linear programming problem. Take pairs of stable marriage payoff functions $(V_m, V_w), (V_m', V_w') \in V(\mu_m, \mu_w)$, then for any $\lambda \in (0, 1)$, $(\lambda V_m + (1-\lambda)V_m', \lambda V_w + (1-\lambda)V_w') \in \Gamma_V(\mu_m, \mu_w)$ because the pair of stable marriage payoff function is also the solution to the linear programming problem. Therefore, $\Gamma_V(\mu_m, \mu_w)$ is convex. Showing compactness of $\Gamma_V(\Gamma_\mu(\Gamma_\sigma(V_m, V_w))) = \Gamma_V(\mu_m, \mu_w)$ follows similar steps of showing the compactness of the space of stable marriage payoff functions.

1.7.3 Proof of Theorem 2

To simplify notations, for $k_1, k_2, k_3 \in \{L, H\}$, let $V_{k_1} = V_m(Y_{mk_1})$, $V_{k_2k_3} = V_w(Y_{wk_2}, R_{k_3})$, $S_{k_1k_2k_3} = S(Y_{mk_1}, Y_{wk_2}, R_{k_3})$.

Equilibrium investments are then

$$\theta_m^* = \frac{C_m}{Y_{mH} - Y_{mL} + V_H^* - V_L^*} \tag{1.3}$$

$$\theta_{w1}^* = \frac{C_w}{Y_{wH} - Y_{wL} + V_{HH}^* - V_{LH}^*} \tag{1.4}$$

$$\theta_{w2}^* = \frac{C_w + \phi_L(V_{LH}^* - V_{LL}^*)}{Y_{wH} - Y_{wL} + \phi_H(V_{HH}^* - V_{LH}^*) + \phi_L(V_{HL}^* - V_{LL}^*)} \tag{1.5}$$

Equilibrium marriage characteristics distributions are

$$\mu_H^* = \int_{\theta_m^*}^1 \theta_m(2 - \theta_m)dF_m(\theta_m) \tag{1.6}$$

$$\mu_L^* = \int_{\theta_m^*}^1 (1 - \theta_m)^2 dF_m(\theta_m) \tag{1.7}$$

$$\mu_{HH}^* = \int_{\theta_{w1}^*}^1 \theta_w dF_w(\theta_w) + \int_{\theta_{w2}^*}^1 \phi_H(1 - \theta_w)\theta_w dF_w(\theta_w) \tag{1.8}$$

$$\mu_{LH}^* = F_w(\theta_{w1}) + \int_{\theta_{w1}^*}^{\theta_{w2}^*} (1 - \theta_w)dF_w(\theta_w) + \int_{\theta_{w2}^*}^1 \phi_H(1 - \theta_w)^2 dF_w(\theta_w) \tag{1.9}$$

$$\mu_{HL}^* = \int_{\theta_{w2}^*}^1 \phi_L\theta_w(2 - \theta_w)dF_w(\theta_w) \tag{1.10}$$

$$\mu_{LL}^* = 1 - \int_{\theta_{w2}^*}^1 \phi_L(1 - \theta_w)^2 dF_w(\theta_w) \tag{1.11}$$

First, suppose that Condition 1.2 holds ($S_{HLH} + S_{LHL} > S_{HHL} + S_{LLH}$). The left

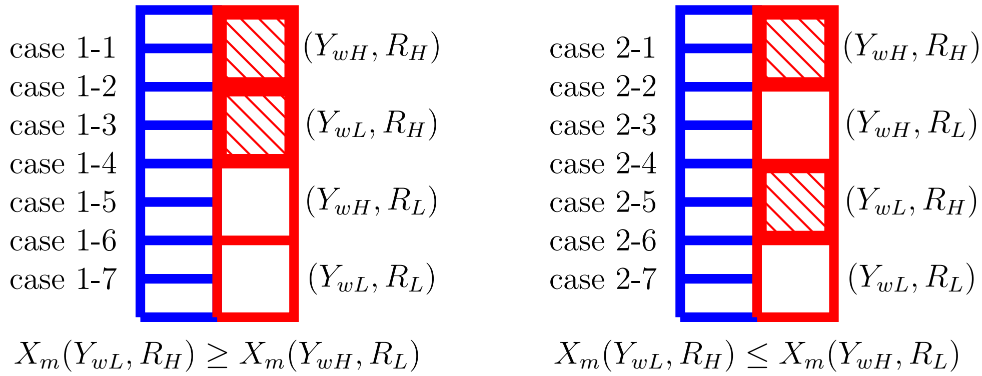

Figure 1.9: Fourteen cases of marriage characteristics distributions and stable matchings when the marriage characteristics take binary values.

panel of Figure 1.3 illustrates the possible equilibrium matching: (Y_{wL}, R_H) women marry higher-income husbands than (Y_{wH}, R_L) women. The equilibrium marriage characteristics distributions can be categorized into the following seven scenarios depending on the relative abundance of high-income men: (1) $\mu^*_H < \mu^*_{HH}$, (2) $\mu^*_H = \mu^*_{HH}$, (3) $\mu^*_H < \mu^*_{HH} + \mu^*_{LH}$, (4) $\mu^*_H = \mu^*_{HH} + \mu^*_{LH}$, (5) $\mu^*_H < \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$, (6) $\mu^*_H = \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$, (7) $\mu^*_H > \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$. The following seven cases are mutually exclusive and comprehensive, under each case equilibrium marriage characteristics distributions are different. Equilibrium marriage payoffs in each case are uniquely determined up to a constant. Equilibrium investments $(\theta^*_m, \theta^*_{w1}, \theta^*_{w2})$ (by equations (1.3)-(1.5)), marriage characteristics distributions (μ^*_m, μ^*_w) (by equations (1.6)-(1.11)) and matching μ^* are uniquely determined.

Case 1-1:

$$
\begin{aligned}
& \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHH}-S_{LHH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m) \\
< \; & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{LHH}-S_{LLH}}}^{1} \theta_w dF_w(\theta_w) \\
& + \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_H(1-\theta_w)\theta_w dF_w(\theta_w).
\end{aligned}
$$

Equilibrium distribution satisfies $\mu^*_H < \mu^*_{HH}$. Equilibrium marriage payoffs are determined by

$$V_H^* - V_L^* = S_{HHH} - S_{LHH}$$

$$V_{HH}^* - V_{LH}^* = S_{LHH} - S_{LLH}$$

$$V_{LH}^* - V_{LL}^* = S_{LLH} - S_{LLL}$$

$$V_{HL}^* - V_{LL}^* = S_{LHL} - S_{LLL}.$$

Case 1-2

$$\begin{aligned}
& \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHH}-S_{LHH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m) \\
\geq & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{LHH}-S_{LLH}}}^{1} \theta_w dF_w(\theta_w) \\
& + \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_H(1-\theta_w)\theta_w dF_w(\theta_w)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLH}-S_{LLH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m) \\
\geq & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{HHH}-S_{HLH}}}^{1} \theta_w dF_w(\theta_w) \\
& + \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_H(1-\theta_w)\theta_w dF_w(\theta_w).
\end{aligned}$$

Equilibrium distribution satisfies $\mu_H^* = \mu_{HH}^*$. Equilibrium marriage payoffs are determined by

$$V_H^* - V_L^* = \lambda^*(S_{HLH} - S_{LLH}) + (1-\lambda^*)(S_{HHH} - S_{LHH})$$

$$V_{HH}^* - V_{LH}^* = \lambda^*(S_{HHH} - S_{HLH}) + (1-\lambda^*)(S_{LHH} - S_{LLH})$$

$$V_{LH}^* - V_{LL}^* = S_{LLH} - S_{LLL}$$

$$V_{HL}^* - V_{LL}^* = S_{LHL} - S_{LLL}.$$

where λ^* is the unique solution to

$$\begin{aligned}
& \int_{\frac{C_m}{Y_{mH}-Y_{mL}+[\lambda(S_{HLH}-S_{LLH})+(1-\lambda)(S_{HHH}-S_{LHH})]}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m) \\
= & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+[\lambda(S_{HHH}-S_{HLH})+(1-\lambda)(S_{LHH}-S_{LLH})]}}^{1} \theta_w dF_w(\theta_w) \\
& + \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H[\lambda(S_{HHH}-S_{HLH})+(1-\lambda)(S_{LHH}-S_{LLH})]+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_H(1-\theta_w)\theta_w dF_w(\theta_w).
\end{aligned}$$

Case 1-3

$$\begin{aligned}
& \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{HHH}-S_{HLH}}}^{1} \theta_w dF_w(\theta_w) \\
& + \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_H(1-\theta_w)\theta_w dF_w(\theta_w) \\
< & \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLH}-S_{LLH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m) \\
< & \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} [\theta_w+\phi_H(1-\theta_w)]dF_w(\theta_w) \\
& + F_w\left(\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{LHL}-S_{LLL})}\right)
\end{aligned}$$

Equilibrium distribution satisfies $\mu^*_{HH} < \mu^*_H < \mu^*_{HH} + \mu^*_{LH}$. Equilibrium payoffs satisfy

$$V^*_H - V^*_L = S_{HLH} - S_{LLH}$$

$$V^*_{HH} - V^*_{LH} = S_{HHH} - S_{HLH}$$

$$V^*_{LH} - V^*_{LL} = S_{LLH} - S_{LLL}$$

$$V^*_{HL} - V^*_{LL} = S_{LHL} - S_{LLL}.$$

Case 1-4

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLH}-S_{LLH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$

$$\geq \quad + \int_{\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} [\theta_w + \phi_H(1-\theta_w)]dF_w(\theta_w)$$
$$+F_w(\frac{C_w+\phi_L(S_{LHL}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{LHL}-S_{LLL})})$$

and

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHL}-S_{LHL}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$
$$\leq \int_{\frac{C_w+\phi_L(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} [\theta_w + \phi_H(1-\theta_w)]dF_w(\theta_w)$$
$$+F_w\left(\frac{C_w+\phi_L(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}\right)$$

Equilibrium distribution satisfies $\mu^*_H = \mu^*_{HH} + \mu^*_{LH}$. Equilibrium marriage payoffs are determined by

$$V^*_H - V^*_L = \lambda^*(S_{HHL}-S_{LHL}) + (1-\lambda^*)(S_{HLH}-S_{LLH})$$

$$V^*_{HH} - V^*_{LH} = S_{HHH} - S_{HLH}$$

$$V^*_{LH} - V^*_{LL} = \lambda^*(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL}) + (1-\lambda^*)(S_{LLH}-S_{LLL})$$

$$V^*_{HL} - V^*_{LL} = S_{LHL} - S_{LLL}$$

where λ^* is the unique solution to

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+[\lambda(S_{HHL}-S_{LHL})+(1-\lambda)(S_{HLH}-S_{LLH})]}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$
$$= \int_{\frac{C_w+\phi_L[\lambda(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})+(1-\lambda)(S_{LLH}-S_{LLL})]}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} [\theta_w + \phi_H(1-\theta_w)]dF_w(\theta_w)$$
$$+F_w\left(\frac{C_w+\phi_L[\lambda(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})+(1-\lambda)(S_{LLH}-S_{LLL})]}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}\right)$$

Case 1-5

$$+\int_{\frac{C_w+\phi_L(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} [\theta_w + \phi_H(1-\theta_w)]dF_w(\theta_w)$$

$$+F_w\left(\frac{C_w+\phi_L(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}\right)$$

$$< \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHL}-S_{LHL}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$

$$< 1-\int_{\frac{C_w+\phi_L(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_L(1-\theta_w)^2dF_w(\theta_w).$$

Equilibrium distribution satisfies $\mu^*_{HH}+\mu^*_{LH} < \mu^*_H < \mu^*_{HH}+\mu^*_{LH}+\mu^*_{HL}$. Equilibrium marriage payoffs are determined by

$$V^*_H - V^*_L = S_{HHL}-S_{LHL}$$

$$V^*_{HH} - V^*_{LH} = S_{HHH}-S_{HLH}$$

$$V^*_{LH} - V^*_{LL} = S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL}$$

$$V^*_{HL} - V^*_{LL} = S_{LHL}-S_{LLL}$$

Case 1-6

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHL}-S_{LHL}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$

$$\geq 1-\int_{\frac{C_w+\phi_L(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_L(1-\theta_w)^2dF_w(\theta_w)$$

and

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLL}-S_{LLL}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$

$$\leq 1-\int_{\frac{C_w+\phi_L(S_{HLH}-S_{HLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{HHL}-S_{HLL})}}^{1} \phi_L(1-\theta_w)^2dF_w(\theta_w).$$

Equilibrium distribution satisfies $\mu^*_H = \mu^*_{HH}+\mu^*_{LH}+\mu^*_{HL}$. Equilibrium marriage payoffs are determined by

$$V^*_H - V^*_L = \lambda^*(S_{HLL}-S_{LLL}) + (1-\lambda^*)(S_{HHL}-S_{LHL})$$

$$V^*_{HH} - V^*_{LH} = S_{HHH} - S_{HLH}$$

$$V^*_{LH} - V^*_{LL} = \lambda^*(S_{HLH} - S_{LLL} - S_{HHL} + S_{LHL}) + (1-\lambda^*)(S_{HLH} - S_{HLL})$$

$$V^*_{HL} - V^*_{LL} = \lambda^*(S_{LHL} - S_{LLL}) + (1-\lambda^*)(S_{HHL} - S_{HLL})$$

where λ^* is the unique solution to

$$\begin{aligned} & \int^1_{\frac{C_m}{Y_{mH}-Y_{mL}+[\lambda(S_{HLL}-S_{LLL})+(1-\lambda)(S_{HHL}-S_{LHL})]}} \theta_m(2-\theta_m)dF_m(\theta_m) \\ = \ & 1 - \int^1_{\frac{C_w+\phi_L[\lambda(S_{HLH}-S_{LLL}-S_{HHL}+S_{LHL})+(1-\lambda)(S_{HLH}-S_{HLL})]}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L[\lambda(S_{LHL}-S_{LLL})+(1-\lambda)(S_{HHL}-S_{HLL})]}} \phi_L(1-\theta_w)^2 dF_w(\theta_w). \end{aligned}$$

Case 1-7

$$\begin{aligned} & \int^1_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLL}-S_{LLL}}} \theta_m(2-\theta_m)dF_m(\theta_m) \\ > \ & 1 - \int^1_{\frac{C_w+\phi_L(S_{HLH}-S_{HLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{HHL}-S_{HLL})}} \phi_L(1-\theta_w)^2 dF_w(\theta_w). \end{aligned}$$

Equilibrium distribution satisfies $\mu^*_H > \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$. Equilibrium marriage payoffs are determined by

$$V^*_H - V^*_L = S_{HLL} - S_{LLL}$$

$$V^*_{HH} - V^*_{LH} = S_{HHH} - S_{HLH}$$

$$V^*_{LH} - V^*_{LL} = S_{HLH} - S_{HLL}$$

$$V^*_{HL} - V^*_{LL} = S_{HHL} - S_{HLL}.$$

Second, suppose that Condition 1.2 does not hold, i.e. $S_{HLH} + S_{LHL} \leq S_{HHL} + S_{LLH}$. In the equilibrium matching, almost all (Y_{wL}, R_H) women marry lower-income husbands than (Y_{wH}, R_L) women do. There are also seven different scenarios depending on equilibrium matching distributions: (1) $\mu^*_H < \mu^*_{HH}$, (2) $\mu^*_H = \mu^*_{HH}$, (3) $\mu^*_{HH} < \mu^*_H < \mu^*_{HH} + \mu^*_{LH}$, (4) $\mu^*_H = \mu^*_{HH} + \mu^*_{LH}$, (5) $\mu^*_{HH} + \mu^*_{LH} < \mu^*_H < \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$, (6) $\mu^*_H = \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$, (7) $\mu^*_H > \mu^*_{HH} + \mu^*_{LH} + \mu^*_{HL}$.

Case 2-1 = Case 1-1

Case 2-2 = Case 1-2

Case 2-3

$$\begin{aligned} & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{LHH}-S_{LLH}}}^{1} \theta_w dF_w(\theta_w) \\ & + \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} \phi_H(1-\theta_w)\theta_w dF_w(\theta_w) \\ < & \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHL}-S_{LHL}}}^{1} \theta_m(2-\theta_m) dF_m(\theta_m) \\ < & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{LHH}-S_{LLH}}}^{1} \theta_w dF_w(\theta_w) \\ & + \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} (1-\theta_w)\theta_w dF_w(\theta_w). \end{aligned}$$

Equilibrium distribution satisfies $\mu_{HH}^* < \mu_H^* < \mu_{HH}^* + \mu_{HL}^*$. Equilibrium marriage payoff functions are determined by

$$V_H^* - V_L^* = S_{HHL} - S_{LHL}$$

$$V_{HH}^* - V_{LH}^* = S_{LHH} - S_{LLH}$$

$$V_{LH}^* - V_{LL}^* = S_{LLH} - S_{LLL}$$

$$V_{HL}^* - V_{LL}^* = S_{LHL} - S_{LLL}.$$

Case 2-4

$$\begin{aligned} & \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{LHH}-S_{LLH}}}^{1} \theta_w dF_w(\theta_w) \\ & + \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{LHH}-S_{LLH})+\phi_L(S_{LHL}-S_{LLL})}}^{1} (1-\theta_w)\theta_w dF_w(\theta_w) \\ \leq & \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HHL}-S_{LHL}}}^{1} \theta_m(2-\theta_m) dF_m(\theta_m) \end{aligned}$$

and

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLH}-S_{LLH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$
$$\leq \int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{HHH}-S_{HLH}}}^{1} \theta_w dF_w(\theta_w) + \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{HHL}-S_{LLL}-S_{HLH}+S_{LLH})}}^{1} (1-\theta_w)\theta_w dF_w(\theta_w).$$

Equilibrium distribution is $\mu^*_{HH} + \mu^*_{HL} = \mu^*_H$. Equilibrium marriage payoffs are determined by

$$V^*_H - V^*_L = \lambda^*(S_{HHL} - S_{LHL}) + (1-\lambda^*)(S_{HLH} - S_{LLH})$$

$$V^*_{HH} - V^*_{LH} = \lambda^*(S_{LHH} - S_{LLH}) + (1-\lambda^*)(S_{HHH} - S_{HLH})$$

$$V^*_{LH} - V^*_{LL} = S_{LLH} - S_{LLL}$$

$$V^*_{HL} - V^*_{LL} = \lambda^*(S_{LHL} - S_{LLL}) + (1-\lambda^*)(S_{HHL} - S_{LLL} - S_{HLH} + S_{LLH})$$

where λ^* is the unique solution to

$$\int_{\frac{C_m}{Y_{mH}-Y_{mL}+\lambda(S_{HHL}-S_{LHL})+(1-\lambda)(S_{HLH}-S_{LLH})}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$
$$= \int_{\frac{C_w}{Y_{wH}-Y_{wL}+\lambda(S_{LHH}-S_{LLH})+(1-\lambda)(S_{HHH}-S_{HLH})}}^{1} \theta_w dF_w(\theta_w) + \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H[\lambda(S_{LHH}-S_{LLH})+(1-\lambda)(S_{HHH}-S_{HLH})] + \phi_L[\lambda(S_{LHL}-S_{LLL})+(1-\lambda)(S_{HHL}-S_{LLL}-S_{HLH}+S_{LLH})]}}^{1} (1-\theta_w)\theta_w dF_w(\theta_w).$$

Case 2-5

$$\int_{\frac{C_w}{Y_{wH}-Y_{wL}+S_{HHH}-S_{HLH}}}^{1} \theta_w dF_w(\theta_w) + \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{HHL}-S_{LLL}-S_{HLH}+S_{LLH})}}^{1} (1-\theta_w)\theta_w dF_w(\theta_w)$$

$$< \int_{\frac{C_m}{Y_{mH}-Y_{mL}+S_{HLH}-S_{LLH}}}^{1} \theta_m(2-\theta_m)dF_m(\theta_m)$$

$$< 1 - \int_{\frac{C_w+\phi_L(S_{LLH}-S_{LLL})}{Y_{wH}-Y_{wL}+\phi_H(S_{HHH}-S_{HLH})+\phi_L(S_{HHL}-S_{LLL}-S_{HLH}+S_{LLH})}}^{1} \phi_L(1-\theta_w)^2 dF_w(\theta_w).$$

Equilibrium distribution is $\mu^*_{HH} + \mu^*_{HL} < \mu^*_H < 1 - \mu^*_{LL}$. Equilibrium marriage payoffs satisfy

$$V^*_H - V^*_L = S_{HLH} - S_{LLH}$$

$$V^*_{HH} - V^*_{LH} = S_{HHH} - S_{HLH}$$

$$V^*_{LH} - V^*_{LL} = S_{LLH} - S_{LLL}$$

$$V^*_{HL} - V^*_{LL} = S_{HHL} - S_{LLL} - S_{HLH} + S_{LLH}.$$

Case 2-6 = Case 1-6

Case 2-7 = Case 1-7

1.7.4 Proof of Proposition 1

When $P_{mH} = P_{wH} \equiv P_H$, $P_{mL} = P_{wL} \equiv P_L$, $F_m = F_w \equiv F$, and $c_m = c_w \equiv c$, the three equilibrium cutoffs become

$$\theta^*_m = \frac{C}{Y_H - Y_L + V^*_m(Y_H) - V^*_m(Y_L)},$$

$$\theta^*_{w1} = \frac{C}{Y_H - Y_L + V^*_w(Y_H, R_H) - V^*_w(Y_L, R_H)},$$

$$\theta^*_{w2} = \frac{C + \phi_L[V^*_w(Y_L, R_L) - \mathbb{E}V^*_w(Y_L, R_L)]}{Y_H - Y_L + \phi_H[V^*_w(Y_H, R_H) - V^*_w(Y_L, R_H)] + \phi_L[V^*_w(Y_H, R_L) - V^*_w(Y_L, R_L)]}.$$

The proof is by contradiction. Suppose that weakly more men go to college than women in equilibrium: $1 - F(\theta^*_m) \geq 1 - F(\theta^*_{w1})$. Hence, $\theta^*_m \leq \theta^*_{w1}$. First, from the equilibrium investment cutoffs determination, $\theta^*_m \leq \theta^*_{w1}$ implies

$$V^*_m(Y_H) - V^*_m(Y_L) \geq V^*_w(Y_H, R_H) - V^*_w(Y_L, R_H) \tag{1.12}$$

Second, since $\theta^*_{w2} > \theta^*_{w1}$, $\theta^*_m \leq \theta^*_{w1} < \theta^*_{w2}$. The mass of high-income men and women are respectively

$$\mu^*_m(Y_H) = \int_{\theta^*_m}^{1} [\theta + (1-\theta)\theta] dF(\theta)$$

$$\mu^*_w(Y_H, R_H) + \mu^*_w(Y_H, R_L) = \int_{\theta^*_{w1}}^{1} \theta dF(\theta) + \int_{\theta^*_{w2}}^{1} (1-\theta)\theta dF(\theta)$$

Hence, more men than women draw from high-income distribution in equilibrium, $\mu^*_m(Y_H) > \mu^*_w(Y_H, R_H) + \mu^*_w(Y_H, R_L)$. Regardless of the equilibrium matching (case 1-5, 1-6, 1-7, 2-5, 2-6, or 2-7 in the equilibrium uniqueness proof), the following stability conditions hold: there are positive measures of matches between Y_H men and (Y_H, R_H) women,

$$V^*_m(Y_H) + V^*_w(Y_H, R_H) = S(Y_H, Y_H, R_H), \tag{1.13}$$

there are positive measures of matches between Y_H men and (Y_H, R_L) women because there are more high-income men than high-income women, so some high-income men marry fit lower-income women,

$$V^*_m(Y_H) + V^*_w(Y_L, R_H) = S(Y_H, Y_L, R_H), \tag{1.14}$$

and finally by the stability condition,

$$V^*_m(Y_L) + V^*_w(Y_L, R_H) \geq S(Y_L, Y_L, R_H). \tag{1.15}$$

Subtract equation (1.14) from equation (1.13),

$$V^*_w(Y_H, R_H) - V^*_w(Y_L, R_H) = S(Y_H, Y_H, R_H) - S(Y_H, Y_L, R_H).$$

Subtract equation (1.15) from equation (1.14),

$$V^*_m(Y_H) - V^*_m(Y_L) \leq S(Y_H, Y_L, R_H) - S(Y_L, Y_L, R_H).$$

By strict supermodularity,

$$\begin{aligned} V_m^*(Y_H) - V_m^*(Y_L) &\leq S(Y_H, Y_L, R_H) - S(Y_L, Y_L, R_H) \\ &< S(Y_H, Y_H, R_H) - S(Y_H, Y_L, R_H) = V_w^*(Y_H, R_H) - V_w^*(Y_L, R_H) \end{aligned}$$

which contradicts inequality (1.12). Therefore, when the surplus function is strictly supermodular in incomes, strictly more women than men go to college.

Suppose that the surplus function is strictly submodular instead, weakly more women than men go to college. In any stable matching, by Lemma 1, almost all (Y_L, R_H) women marry higher-income husbands than (Y_H, R_H) and (Y_L, R_L) women, and almost all (Y_L, R_L) women marry higher-income husbands than (Y_H, R_L) women. Therefore, at least some (Y_L, R_H) women marry the highest-income husbands. By the stability condition, $V_m^*(Y_H)+V_w^*(Y_L, R_H) = S(Y_H, Y_L, R_H)$ where equality follows from symmetry. By the stability condition for Y_L men and (Y_H, R_H), $V_m^*(Y_L) + V_w^*(Y_H, R_H) \geq S(Y_L, Y_H, R_H) = S(Y_H, Y_L, R_H)$. The two stability conditions combine to

$$V_m^*(Y_H) - V_m^*(Y_L) \leq V_w^*(Y_L, R_H) - V_w^*(Y_H, R_H).$$

As a result, $\theta_m^* \geq \theta_{w1}^*$, and $1 - F_m(\theta_m^*) \leq 1 - F_w(\theta_{w1}^*)$. The inequality holds strictly when there is no match between Y_L men and (Y_H, R_H) women, and strictly more women than men go to college.

1.7.5 Extended Model and Proof of Proposition 5

Suppose that an agent can enter the marriage market and invest in the same period. Assume that their subsequent investment plans are publicly known and they can commit to these plans. An age 1 agent can choose among four actions: to go to college and to delay marriage (I_1, D_1), to go to college and to enter the marriage market in the current period (I_1, E_1), not to invest and to delay marriage (N_1, D_1), and not to invest and to enter the marriage market (N_1, E_1). Easily we can see that (N_1, D_1) is strictly dominated by (N_1, E_1) because there is no gain in delaying marriage if an agent does not go to college. An age 2 agent chooses among four strategies: to make a career investment and to delay marriage (I_2, D_2), to make a career investment and to enter the marriage market (I_2, E_2), not to make a career investment and to delay marriage (N_2, D_2), not to make a career investment and to enter the marriage market (N_2, E_2). (N_2, D_2) is dominated by (N_2, E_2).

A man who chooses (N_1, E_1) or (N_2, E_2) enters the marriage market with marriage type Y_{m1} or Y_{m2}. A man who chooses (I_1, E_1) enters the marriage market at age 1 but does not have his income realized. His marriage type is $\sigma_{m2}(\theta_m, \cdot)$ where $\sigma_{m2}(\theta_m, Y_{m2})$ is his probability of investing when he receives an income offer Y_{m2}. A man who chooses (I_2, D_2) enters the marriage market in the second period, but does not have his income realized. His marriage type is $P_{m3}(\cdot|\theta_m)$, the distribution he draws his income from in the third period. In summary, a man's marriage type can be Y_m, $\sigma_{m2}(\theta_m, \cdot)$, or $P_{m3}(\cdot|\theta_m)$. Similarly, a woman's marriage type is (Y_w, R), $\sigma_{w2}(\theta_w, \cdot)$, or $P_{w3}(\cdot|\theta_w)$.

Let X_m and X_w denote a man's and a woman's marriage characteristics, and $\mathcal{X}_m$ and $\mathcal{X}_w$ the set of men's and women's characteristics, respectively. Let $\widetilde{S}(X_m, X_w)$ denote the expected marriage surplus of a X_m man and a X_w woman. Assume that a man and a woman start to receive their marriage surplus when both of them realize their income. Therefore, the marriage surplus function is

$$\widetilde{S}(Y_m, Y_w, R) = S(Y_m, Y_w, R)$$

$$\widetilde{S}(Y_m, P_{w3}(\cdot|\theta_w)) = \delta \mathbb{E}_{Y_{w3},R}[S(Y_m, Y_{w3}, R)|\theta_w]$$

$$\begin{aligned} \widetilde{S}(Y_m, \sigma_{w2}(\theta_w, \cdot)) &= \delta \mathbb{E}_{Y_{w2}}[(1 - \sigma_{w2}(\theta_w, Y_{w2}))S(Y_m, Y_{w2}, R_H) \\ &\quad + \sigma_{w2}(\theta_w, Y_{w2})\widetilde{S}(Y_m, P_{w3}(\cdot|\theta_w))|\theta_w] \end{aligned}$$

$$\widetilde{S}(P_{m3}(\cdot|\theta_m), Y_w, R) = \delta\mathbb{E}_{Y_{m3}}[S(Y_{m3}, Y_w, R)|\theta_m]$$

$$\widetilde{S}(P_{m3}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w)) = \delta\mathbb{E}_{Y_{m3},Y_{w3},R}[S(Y_{m3}, Y_{w3}, R)|\theta_m, \theta_w]$$

$$\begin{aligned}\widetilde{S}(P_{m3}(\cdot|\theta_m), \sigma_{w2}(\theta_w, \cdot)) &= \delta\mathbb{E}_{Y_{m3},Y_{w2}}[(1-\sigma_{w2}(\theta_w, Y_{w2}))S(Y_{m3}, Y_{w2}, R_H) \\ &\quad +\sigma_{w2}(\theta_w, Y_{w2})\widetilde{S}(Y_{m3}, P_{w3}(\cdot|\theta_w))|\theta_m, \theta_w]\end{aligned}$$

$$\begin{aligned}\widetilde{S}(\sigma_{m2}(\theta_m, \cdot), Y_w, R) &= \delta\mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m, Y_{m2}))S(Y_{m2}, Y_w, R) \\ &\quad +\sigma_{m2}(\theta_m, Y_{m2})\widetilde{S}(P_{m3}(\cdot|\theta_m), Y_w, R)|\theta_m]\end{aligned}$$

$$\begin{aligned}\widetilde{S}(\sigma_{m2}(\theta_m, \cdot), P_{w3}(\cdot|\theta_w)) &= \delta\mathbb{E}_{Y_{m2},Y_{w3},R}[(1-\sigma_{m2}(\theta_m, Y_{m2}))S(Y_{m2}, Y_{w3}, R) \\ &\quad +\sigma_{m2}(\theta_m, Y_{m2})\widetilde{S}(P_{m3}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w))|\theta_m, \theta_w]\end{aligned}$$

$$\begin{aligned}\widetilde{S}(\sigma_{m2}(\theta_m, \cdot), \sigma_{w2}(\theta_w, \cdot)) &= \delta\mathbb{E}_{Y_{m2},Y_{w2}}[(1-\sigma_{m2}(\theta_m, Y_{m2}))(1-\sigma_{w2}(\theta_w, Y_{w2}))S(Y_{m2}, Y_{w2}, R_H) \\ &\quad +(1-\sigma_{m2}(\theta_m, Y_{m2}))\sigma_{w2}(\theta_w, Y_{w2})\widetilde{S}(Y_{m2}, P_{w3}(\cdot|\theta_w)) \\ &\quad +\sigma_{m2}(\theta_m, Y_{m2})(1-\sigma_{w2}(\theta_w, Y_{w2}))\widetilde{S}(P_{m2}(\cdot|\theta_m), Y_{w2}, R_H) \\ &\quad +\sigma_{m2}(\theta_m, Y_{m2})\sigma_{w2}(\theta_w, Y_{w2})\widetilde{S}(P_{m2}(\cdot|\theta_m), P_{w2}(\cdot|\theta_w))|\theta_m, \theta_w].\end{aligned}$$

Let $\widetilde{\mu}_m$ and $\widetilde{\mu}_w$ represent respectively measures of men's and women's marriage characteristics. Let $\widetilde{\mu}$ be the matching measure on the expanded marriage characteristics set $\mathcal{X}_m \times \mathcal{X}_w$. Let $\widetilde{V}_m : \mathcal{X}_m \to \mathbb{R}$ and $\widetilde{V}_w : \mathcal{X}_w \to \mathbb{R}$ be men's and women's marriage payoff functions. A marriage market outcome $(\widetilde{\mu}, \widetilde{V}_m, \widetilde{V}_w)$ is stable if $\widetilde{\mu}$ has marginals $\widetilde{\mu}_m$ and $\widetilde{\mu}_w$, $\widetilde{V}_m(\delta P_m) \geq 0$ and $\widetilde{V}_m(X_m) \geq 0$ for all X_m, $\widetilde{V}_w(X_w) \geq 0$ for all X_w, $\widetilde{V}_m(X_m) + \widetilde{V}_w(X_w) = \widetilde{S}(X_m, X_w)$ for all $\forall(X_m, X_w) \in \text{supp}(\widetilde{\mu})$, and $\widetilde{V}_m(X_m) + \widetilde{V}_w(X_w) \geq \widetilde{S}(X_m, X_w)$ for all $X_m \in \text{supp}(\widetilde{\mu}_m)$ and $X_w \in \text{supp}(\widetilde{\mu}_w)$.[8]

[8]See Borch (1962); Wilson (1968); Chiappori and Reny (2015); Zhang (2015c) for settings with income uncertainty.

Now consider an age 2 ability θ_m man who decides between investing and marrying (I_2, E_2), and investing and delaying marriage (I_2, D_2). The marriage timing decision does not affect the labor market return or the investment cost, the only difference is in the marriage market payoff. If he invests and enters the marriage market, his marriage type is $P_{m3}(\cdot|\theta_m)$, and his marriage market payoff is $\widetilde{V}_m(P_{m3}(\cdot|\theta_m))$. If he invests and delays marriage, he enters the marriage market in the third period, and his expected payoff is $\delta\mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m]$. It turns out that investing and delaying marriage always dominates investing and marrying at the same time. Consider the following logic. Suppose that the $P_{m3}(\cdot|\theta_m)$ is matched to a (Y_w, R) woman in the marriage market, so they divide up their marriage surplus,

$$\widetilde{S}(P_{m3}(\cdot|\theta_m), Y_w, R) = \widetilde{V}_m(P_{m3}(\cdot|\theta_m)) + \widetilde{V}_w(Y_w, R).$$

A $P_{m3}(\cdot|\theta_m)$ man's marriage payoff is

$$\begin{aligned} \widetilde{V}_m(P_{m3}(\cdot|\theta_m)) &= \delta\mathbb{E}_{Y_{m3}}[S(Y_{m3}, Y_w, R)|\theta_m] - \widetilde{V}_w(Y_w, R) \\ &= \delta\mathbb{E}_{Y_{m3}}[S(Y_{m3}, Y_w, R) - \widetilde{V}_w(Y_w, R)|\theta_m] - (1-\delta)\widetilde{V}_w(Y_w, R) \end{aligned}$$

By the stability condition, a Y_{m3} man's marriage payoff is weakly higher than the payoff he gets with a (Y_w, R) woman,

$$\widetilde{V}_m(Y_{m3}) \geq S(Y_{m3}, Y_w, R) - \widetilde{V}_w(Y_w, R).$$

Since this inequality is true for any Y_{m3},

$$\widetilde{V}_m(P_{m3}(\cdot|\theta_m)) \leq \delta\mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m] - (1-\delta)\widetilde{V}_w(Y_w, R).$$

Since $\widetilde{V}_w(Y_w, R) > 0$, or $\widetilde{V}_m(Y_{m3}) > S(Y_{m3}, Y_w, R) - \widetilde{V}_w(Y_w, R)$ for some Y_m, $\widetilde{V}_m(P_{m3}(\cdot|\theta_m)) < \delta\mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m]$. Therefore, a man prefers making a career investment and delaying marriage to making a career investment and marrying at the same time if he is matched to a a type Y_w woman. The same preference stays if he is matched to a type $P_{w3}(\cdot|\theta_w)$ or

$\sigma_{w3}(\theta_w, \cdot)$ woman. Furthermore, a man prefers going to college and marrying later over going to college and marrying at the same time.

It has been shown above that when a man who makes a career investment and marries in the same period marries a (Y_w, R) woman, he strictly prefers to delay marriage to the next period. In general, we can show that investing and marrying at the same time is always dominated by investing and delaying marriage when the agent is sufficiently patient. Now we show the preference when he marries a $P_{w3}(\cdot|\theta_w)$ woman. The marriage payoff of investing and marrying at the same time is

$$\begin{aligned}
\widetilde{V}_m(P_{m3}(\cdot|\theta_m)) &= \widetilde{S}(P_{m3}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w)) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)) \\
&= \delta \mathbb{E}_{Y_{m3},Y_{w3},R}[S(Y_{m3}, Y_{w3}, R)|\theta_m, \theta_w] - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)) \\
&= \mathbb{E}_{Y_{m3}}\left[\delta \mathbb{E}_{Y_{w3},R}[S(Y_{m3}, Y_{w3}, R)|\theta_w] - \widetilde{V}_w(P_{w3}(\cdot|\theta_w))|\theta_m\right] \\
&= \mathbb{E}_{Y_{m3}}\left[\widetilde{S}(Y_{m3}, P_{w3}(\cdot|\theta_w))|\theta_w] - \widetilde{V}_w(P_{w3}(\cdot|\theta_w))|\theta_m\right] \\
&< \mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m]
\end{aligned}$$

where the inequality follows from the stability condition for each Y_{m3},

$$\widetilde{S}(Y_{m3}, P_{w3}(\cdot|\theta_w))|\theta_w] - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)) \leq \widetilde{V}_m(Y_{m3}),$$

with inequality holding for almost all Y_{m3}. When δ is sufficiently large, $\delta \mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m] > \widetilde{V}_m(P_{m3}(\cdot|\theta_m))$. Suppose a $P_{m3}(\cdot|\theta_m)$ man marries a $\sigma_{m2}(\theta_m, \cdot)$ woman,

$$\begin{aligned}
\widetilde{V}_m(P_{m3}(\cdot|\theta_m)) &= \widetilde{S}(P_{m3}(\cdot|\theta_m), \sigma_{w2}(\theta_w, \cdot)) - \widetilde{V}_w(\sigma_{w2}(\theta_w, \cdot)) \\
&= \delta \mathbb{E}_{Y_{m3},Y_{w2}}[(1 - \sigma_{w2}(\theta_w, Y_{w2}))S(Y_{m3}, Y_{w2}, R_H) \\
&\qquad + \sigma_{w2}(\theta_w, Y_{w2})\widetilde{S}(Y_{m3}, P_{w3}(\cdot|\theta_w))|\theta_m, \theta_w] - \widetilde{V}_w(\sigma_{w2}(\theta_w, \cdot)) \\
&= \mathbb{E}_{Y_{m3}}[\delta \mathbb{E}_{Y_{w2}}[\widetilde{S}(Y_{m3}, \sigma_{w2}(\theta_w, \cdot)) - \widetilde{V}_w(\sigma_{w2}(\theta_w, \cdot))] \\
&< \mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m].
\end{aligned}$$

When δ is sufficiently large, $\delta \mathbb{E}_{Y_{m3}}[\widetilde{V}_m(Y_{m3})|\theta_m] > \widetilde{V}_m(P_{m3}(\cdot|\theta_m))$. Therefore, we have shown that (I_2, D_2) dominates (I_2, E_2) when a man is sufficiently patient.

We now show similarly that (I_2, D_2) dominates (I_2, E_2) when a woman is sufficiently patient. The marriage payoff of (I_2, D_2) is $\delta \mathbb{E}_{Y_{w3},R}[\widetilde{V}_w(Y_w)|\theta_w]$, and the marriage payoff of (I_2, E_2) is $\widetilde{V}_w(P_{w3}(\cdot|\theta_w))$. If a $P_{w3}(\cdot|\theta_w)$ woman marries a Y_m man, her payoff is

$$\begin{aligned}
\widetilde{V}_w(P_{w3}(\cdot|\theta_w)) &= \widetilde{S}(Y_m, P_{w3}(\cdot|\theta_w)) - \widetilde{V}_m(Y_m) \\
&= \delta \mathbb{E}_{Y_{w3},R}[S(Y_m, Y_{w3}, R)|\theta_w] - \widetilde{V}_m(Y_m) \\
&< \delta \mathbb{E}_{Y_{w3},R}[S(Y_m, Y_{w3}, R) - \widetilde{V}_m(Y_m)|\theta_w] \\
&\leq \delta \mathbb{E}_{Y_{w3},R}[\widetilde{V}_w(Y_w)|\theta_w].
\end{aligned}$$

If a $P_{w3}(\cdot|\theta_w)$ woman marries a $P_{m3}(\cdot|\theta_m)$ man, her payoff is

$$\begin{aligned}
\widetilde{V}_w(P_{w3}(\cdot|\theta_w)) &= \widetilde{S}(P_{m3}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w)) - \widetilde{V}_m(P_{m3}(\cdot|\theta_m)) \\
&= \delta \mathbb{E}_{Y_{m3},Y_{w3},R}[S(Y_{m3}, Y_{w3}, R)|\theta_m, \theta_w] - \widetilde{V}_m(P_{m3}(\cdot|\theta_m)) \\
&= \mathbb{E}_{Y_{w3},R}[\delta \mathbb{E}_{Y_{m2}}[S(Y_m, Y_{w3}, R)|\theta_m] - \widetilde{V}_m(Y_m)|\theta_w] \\
&< \mathbb{E}_{Y_{w3},R}[\widetilde{V}_w(Y_w, R)|\theta_w].
\end{aligned}$$

When δ is sufficiently large, $\widetilde{V}_w(P_{w3}(\cdot|\theta_w)) < \mathbb{E}_{Y_{w3},R}[\widetilde{V}_w(Y_w, R)|\theta_w]$. If a $P_{w3}(\cdot|\theta_w)$ woman marries a $\sigma_{m2}(\theta_m, \cdot)$ man,

$$\begin{aligned}
\widetilde{V}_w(P_{w3}(\cdot|\theta_w)) &= \widetilde{S}(\sigma_{m2}(\theta_m, \cdot), P_{w3}(\cdot|\theta_w)) - \widetilde{V}_m(\sigma_{m2}(\theta_m, \cdot)) \\
&= \delta \mathbb{E}_{Y_{m2},Y_{w3},R}[(1 - \sigma_{m2}(\theta_m, Y_{m2}))S(Y_{m2}, Y_{w3}, R) \\
&\qquad + \sigma_{m2}(\theta_m, Y_{m2})\widetilde{S}(P_{m3}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w))|\theta_m, \theta_w] - \widetilde{V}_m(\sigma_{m2}(\theta_m, \cdot)) \\
&= \mathbb{E}_{Y_{w3},R}[\delta \mathbb{E}_{Y_{m2}}[(1 - \sigma_{m2}(\theta_m, Y_{m2}))S(Y_{m2}, Y_{w3}, R) \\
&\qquad + \sigma_{m2}(\theta_m, Y_{m2})\widetilde{S}(P_{m3}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w)) - \widetilde{V}_m(\sigma_{m2}(\theta_m, \cdot))|\theta_m, \theta_w]] \\
&= \mathbb{E}_{Y_{w3},R}[\delta \mathbb{E}_{Y_{m2}}[\widetilde{S}(\sigma_{m2}(\theta_m, Y_{m2}), Y_w, R) - \widetilde{V}_m(\sigma_{m2}(\theta_m, \cdot))|\theta_m, \theta_w]]
\end{aligned}$$

$$< \quad \mathbb{E}_{Y_{w3},R}[\widetilde{V}_w(Y_w,R)|\theta_w]$$

When δ is sufficiently large, $\widetilde{V}_w(P_{w3}(\cdot|\theta_w)) < \mathbb{E}_{Y_{w3},R}[\widetilde{V}_w(Y_w,R)|\theta_w]$. Therefore, we have shown that (I_2, D_2) dominates (I_2, E_2) when a woman is sufficiently patient.

We next show that (I_1, D_1) dominates (I_1, E_1) for sufficiently patient men. Let $\sigma_{m2}(\theta_m, \cdot)$ be an ability θ_m man's second period investment strategy. If he chooses (I_1, D_1), his expected marriage payoff is

$$\delta\mathbb{E}_{Y_{m2}}\left[(1-\sigma_{m2}(\theta_m,Y_{m2}))\widetilde{V}_m(Y_{m2}) + \sigma_{m2}(\theta_m,Y_{m2})\mathbb{E}_{Y_{m3}}[\delta\widetilde{V}_m(Y_{m3})|\theta_m]|\theta_m\right].$$

If he chooses (I_1, E_1), his expected marriage payoff is $\widetilde{V}_m(\sigma_{m2}(\theta_m,\cdot))$. Suppose that he matches with a (Y_w, R) woman, and they divide their marriage surplus,

$$\begin{aligned}
\widetilde{V}_m(\sigma_{m2}(\theta_m,\cdot)) &= \widetilde{S}(\sigma_{m2}(\theta_m,\cdot),Y_w,R) - \widetilde{V}_w(Y_w,R) \\
&= \delta\mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m,Y_{m2}))S(Y_{m2},Y_w,R) \\
&\quad +\sigma_{m2}(\theta_m,Y_m)\widetilde{S}(P_{m3}(\cdot|\theta_m),Y_w,R)|\theta_m] - \widetilde{V}_w(Y_w,R) \\
&= \mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m,Y_{m2}))(\delta S(Y_{m2},Y_w,R) - \widetilde{V}_w(Y_w,R))|\theta_m] \\
&\quad +\mathbb{E}_{Y_{m2}}[\sigma_{m2}(\theta_m,Y_{m2})(\delta\widetilde{S}(P_{m3}(\cdot|\theta_m),Y_w,R) - \widetilde{V}_w(Y_w,R))|\theta_m] \\
&< \delta\mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m,Y_{m2}))\widetilde{V}_m(Y_{m2})|\theta_m] \\
&\quad +\delta\mathbb{E}_{Y_{m2}}[\sigma_{m2}(\theta_m,Y_{m2})\mathbb{E}_{Y_{m3}}[\delta\widetilde{V}_m(Y_{m3})|\theta_m]|\theta_m].
\end{aligned}$$

The man's marriage payoff $\widetilde{V}_m(\sigma_{m2}(\theta_m,\cdot))$ if he marries with a $P_{w3}(\cdot|\theta_w)$ woman is

$$\begin{aligned}
&\widetilde{S}(\sigma_{m2}(\theta_m,\cdot),P_{w3}(\cdot|\theta_w)) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)) \\
=\ & \mathbb{E}_{Y_{m2}}[\mathbb{E}_{Y_{w3},R}[(1-\sigma_{m2}(\theta_m,Y_{m2}))(\delta S(Y_{m2},Y_{w3},R) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)))|\theta_w]|\theta_m] \\
&+\mathbb{E}_{Y_{m2}}[\mathbb{E}_{Y_{w3},R}[\sigma_{m2}(\theta_m,Y_{m2})(\delta\widetilde{S}(P_{m3}(\cdot|\theta_m),P_{w3}(\cdot|\theta_w)) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)))|\theta_m,\theta_w] \\
\leq\ & \mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m,Y_{m2}))\widetilde{V}_m(Y_{m2}) + \sigma_{m2}(\theta_m,Y_{m2})\mathbb{E}_{Y_{m3}}[\delta\widetilde{V}_m(Y_{m3})|\theta_m]|\theta_m].
\end{aligned}$$

For sufficiently large δ, $\widetilde{V}_m(\sigma_{m2}(\theta_m, \cdot))$ is smaller than

$$\delta \mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m, Y_{m2}))\widetilde{V}_m(Y_{m2}) + \sigma_{m2}(\theta_m, Y_{m2})\mathbb{E}_{Y_{m3}}[\delta \widetilde{V}_m(Y_{m3})|\theta_m]|\theta_m].$$

If a $\sigma_{m2}(\theta_m, \cdot)$ man is matched with a $\sigma_{w2}(\theta_w, \cdot)$ woman, then his marriage payoff $\widetilde{V}_m(\sigma_{m2}(\theta_m,$ $\cdot))$

$$\begin{aligned}
&= \widetilde{S}(\sigma_{m2}(\theta_m, \cdot), \sigma_{w2}(\theta_w, \cdot)) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)) \\
&= \mathbb{E}\left[(1-\sigma_{m2}(\theta_m, Y_{m2}))(1-\sigma_{w2}(\theta_w, Y_{w2}))\left(\delta S(Y_{m2}, Y_{w2}, R_H) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)\right)\right] \\
&\quad + \mathbb{E}\left[(1-\sigma_{m2}(\theta_m, Y_{m2}))\sigma_{w2}(\theta_w, Y_{w2})\left(\delta \widetilde{S}(Y_{m2}, P_{w3}(\cdot|\theta_w)) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)\right)\right] \\
&\quad + \mathbb{E}\left[\sigma_{m2}(\theta_m, Y_{m2})(1-\sigma_{w2}(\theta_w, Y_{w2}))\left(\delta \widetilde{S}(P_{m2}(\cdot|\theta_m), Y_{w2}, R_H) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)\right)\right] \\
&\quad + \mathbb{E}\left[\sigma_{m2}(\theta_m, Y_{m2})\sigma_{w2}(\theta_w, Y_{w2})\left(\delta \widetilde{S}(P_{m2}(\cdot|\theta_m), P_{w3}(\cdot|\theta_w)) - \widetilde{V}_w(P_{w3}(\cdot|\theta_w)\right)\right] \\
&\leq \mathbb{E}\left[(1-\sigma_{m2}(\theta_m, Y_{m2}))(1-\sigma_{w2}(\theta_w, Y_{w2}))\widetilde{V}_m(Y_{m2})\right] \\
&\quad + \mathbb{E}\left[(1-\sigma_{m2}(\theta_m, Y_{m2}))\sigma_{w2}(\theta_w, Y_{w2})\widetilde{V}_m(Y_{m2})\right] \\
&\quad + \mathbb{E}\left[\sigma_{m2}(\theta_m, Y_{m2})(1-\sigma_{w2}(\theta_w, Y_{w2}))\widetilde{V}_m(P_{m2}(\cdot|\theta_m))\right] \\
&\quad + \mathbb{E}\left[\sigma_{m2}(\theta_m, Y_{m2})\sigma_{w2}(\theta_w, Y_{w2})\widetilde{V}_m(P_{m2}(\cdot|\theta_m))\right] \\
&\leq \mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m, Y_{m2}))\widetilde{V}_m(Y_{m2})|\theta_m] \\
&\quad \mathbb{E}_{Y_{m2}}[\sigma_{m2}(\theta_m, Y_{m2})\mathbb{E}_{Y_{m3}}[\delta \widetilde{V}_m(Y_{m3})|\theta_m]|\theta_m]
\end{aligned}$$

where the last inequality follows from the fact that $\widetilde{V}_m(P_{m2}(\cdot|\theta_m)) \leq \mathbb{E}_{Y_{m3}}[\delta \widetilde{V}_m(Y_{m3})|\theta_m]$ for sufficiently large δ. For sufficiently large δ, $\widetilde{V}_m(\sigma_{m2}(\theta_m, \cdot))$ is smaller than

$$\delta \mathbb{E}_{Y_{m2}}[(1-\sigma_{m2}(\theta_m, Y_{m2}))\widetilde{V}_m(Y_{m2}) + \sigma_{m2}(\theta_m, Y_{m2})\mathbb{E}_{Y_{m3}}[\delta \widetilde{V}_m(Y_{m3})|\theta_m]|\theta_m].$$

Therefore, we have shown that no matter who the man is matched to in the marriage market, if he is sufficiently patient, (I_1, E_1) is dominated. We can use analogous arguments to show that (I_1, D_1) dominates (I_1, E_1) for sufficiently patient women.

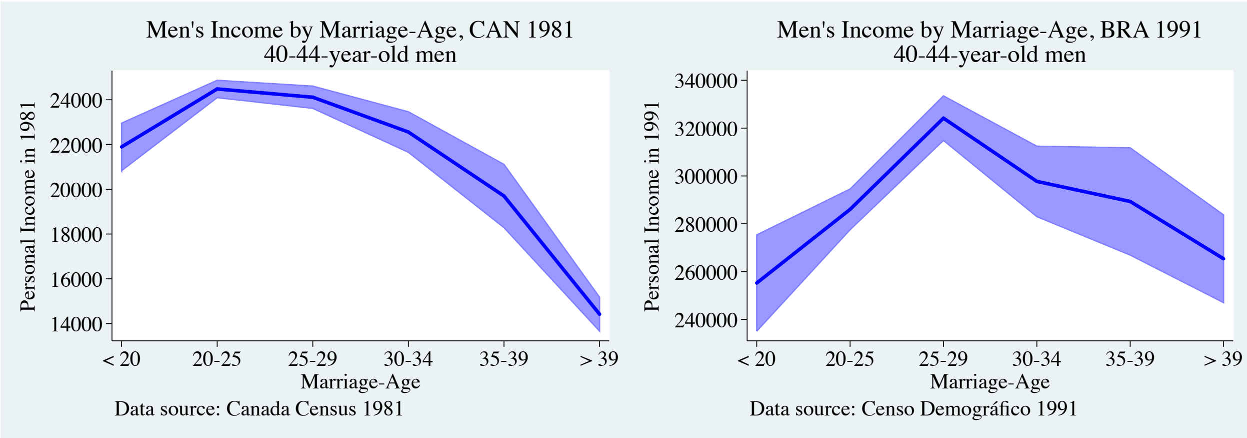

Figure 1.10: The relationships between marriage-age and income among Canadian men ages 40-44 in 1981 and among Brazilian men ages 40-44 in 1991.

1.8 Data Description and Additional Figures

The data in USA 1980 were taken from the 1% sample of the 1980 US Census via IPUMS (Ruggles et al., 2010). Age at first marriage (AGEMARR) and personal income (INCTOT) are directly asked on the form. To be comparable with 2010, I dropped all the people who have married more than once, which constituted 18% of the sample; including those people in the analysis does not change the relationship qualitatively. The data in USA 2012 were taken from the 1% sample of the 2012 American Community Survey via IPUMS-USA (Ruggles et al., 2010). Marriage-age is calculated for those who married once in the current marital status (YRMARR). Those who married more than once constitute about 16% of the sample and are dropped because marriage-age cannot be computed for this subsample.

The Canadian data in 1981 were taken from 1980 Census of Canada via IPUMS (Ruggles et al., 2010). Age at first marriage was directly asked and recorded as AGEMARR. Personal income is INCTOT. The Brazilian men ages 40-44 in 1991 were the the men from 1991 Censo Demogràfico via IPUMS-International (Ruggles et al., 2010). Marriage-age was directly asked and recorded as AGEMARR. Personal income is INCTOT. Questions related to family

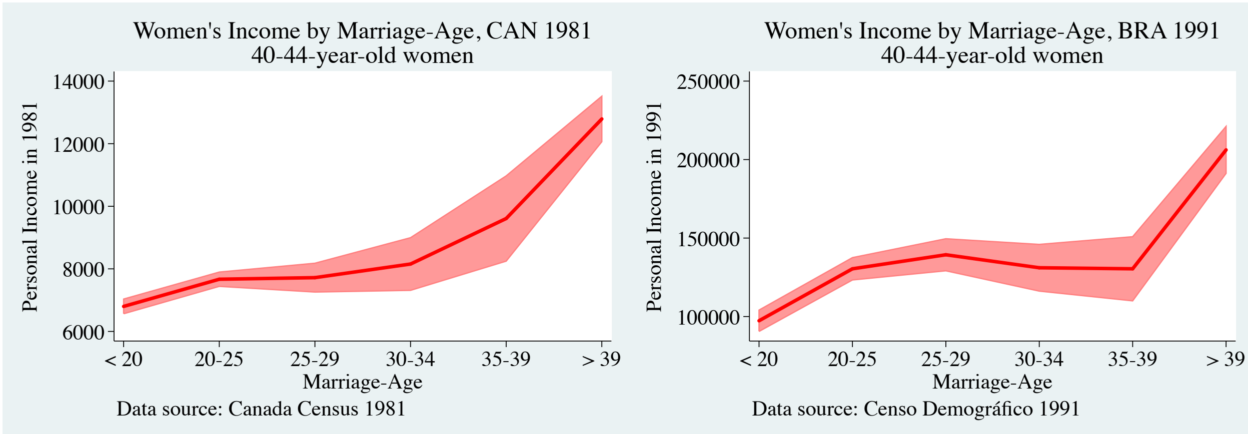

Figure 1.11: The relationship between marriage-age and income among Canadian women ages 40-44 in 1981 and among Brazilian women ages 40-44 in 1991.

income or spouse's income are not asked. I choose these two countries because of data availability: those were the only two countries with data on both personal income and marriage-age in IPUMS-International.

The husband's income is calculated as the family income minus personal income (FTOTINC − INCTOT). Questions related to family income or spouse's income were not asked in the Canadian and Brazilian datasets.

Chapter 2 Pre-Matching Gambles

> "The lottery of the law [profession] ... is very far from being a perfectly fair lottery; and that as well as many other liberal and honorable professions, is, in point of pecuniary gain, evidently under-recompensed. Those professions keep their level, however, with other occupations; and, notwithstanding these discouragements, all the most generous and liberal spirits are eager to crowd into them."
>
> – Adam Smith, Wealth of Nations

Abstract: This paper studies gambling behavior in an equilibrium two-sided matching market. People choose lotteries to change their matching attributes before they enter a stable matching market. These choices can be financial investments, occupation choices, or college major choices. Surprisingly, risk-averse people can take very risky investments before they enter a matching market. A unique fundamental feature of the matching market, I call the stable rematching effect, drives the risky choices: a gamble changes not only a person's matching attribute but also his or her partner. Furthermore, the gambling incentive arises independently of the shape of the surplus function contrary to the possible misconceived intuition that surplus supermodularity and positive-assortative matching resulted from the surplus play a key role. The paper discusses the trade-off between efficiency and inequality: the socially efficient gambling can always be supported in equilibrium, but there may exist other equilibria with different equilibrium wealth distributions. Finally, I interpret pre-matching gambling as choosing careers with different wage distributions and account for gender-specific patterns of occupational choices and marital timing.

Keywords: investment-and-matching, stable rematching effect, income inequality, occupational choice

JEL: C78, D31, J41

2.1 Introduction

People take risks. Ordinary you and I visit casinos and buy lottery tickets. Sane citizens commit crimes and take chance to park illegally. College graduates pick career paths with similar expected lifetime earnings but very different income distributions. Young professionals quit their steady jobs and become entrepreneurs. Investment managers build risky portfolios.

Adam Smith (1776) conjectures that social status motives and overconfidence drive these risky choices. Friedman and Savage (1948) postulate that utility functions exhibit convexity in some range. Rubin and Paul (1979) point out that people may gamble to reach subsistence level (e.g. stealing, robbery). Robson (1992) and Becker et al. (2005) reemphasize the importance of social status. I argue in this paper that subsequent matching concerns induce gambling. College students choose different majors that provide different bundles of human capital in order to meet future employers' needs in the competitive labor market. People risk their wealth in order to attract mates in the future. Investment managers build risky portfolios because financial investment returns not only augment cash flow but also attract future investors.

This paper is the first to study gambling behavior in an equilibrium two-sided matching market. Agents can pick lotteries to change their matching characteristics before they find partners and bargain for their surplus. Studying these often overlooked investments in an equilibrium setting can lead to quite surprising implications about private and social risk-taking. The inherent competitive structure of the matching market in fact encourages gambling.

The paper contains the following key results. First, the paper rationalizes seemingly risky choices before matching markets. Second, the paper discovers a gambling-inducing factor inherent in stable matching market independent of any assumption on the surplus function. Third, the paper investigates the properties of equilibria. Fourth and finally, the paper provides a tractable model that is consistent with key stylized facts about career choices and the marriage market.

Beneficial pre-matching gambles can be prevalent. I first show in an example that an agent may strictly prefer to take an actuarially unfair gamble. Moderate amount of risk aversion does not preclude agents from taking these unfair and risky gambles. The example also alludes to the factors that drive the seemingly risk-seeking preferences.

Stability of the transferable utilities matching market inherently encourages and induces pre-matching risk-taking. A stable rematching effect always prompts the agent to gamble. A gamble does not only change an agent's contribution to surplus but also his or her partner. Crucially, due to stability, for every realization of the gamble, the agent is always matched with the partner that gives him the highest attainable payoff. This stable rematching effect is exclusive to matching market.

The risk-taking behavior, despite of its usual negative connotation, is not necessarily undesirable in the current matching setting. The gambles can be beneficial to private as well as social welfare. The gambles in equilibrium are constrained efficient. That is, if a social planner can redo the gambles everyone on one side of the market takes and rearranges the matches based on the realizations of the gambles, he cannot improve social welfare. When multiple equilibria arise, carefully designed tax schemes can eliminate inefficient equilibrium. I show an example where a progressive tax scheme can effectively enhance efficiency, reduce inequality, and generate positive tax revenue.

I use a dynamic extension of the model to study gender differences in occupational choices and marital patterns. If women remain reproductively desirable for a shorter length of time than men and it takes time to resolve uncertainties from risky investment, women will choose careers that have steady gains and marry early. Men on the other hand choose occupations that may have volatile returns (e.g. traders, entrepreneurs) and marry later in general. As fecundity constraint lessens, more women enter the occupations previously dominated by men and are exposed to labor market shocks and delay their marriages.

After a brief literature review, the rest of the paper proceeds as follows. Section 2.2 demonstrates an example in which an agent could find profitable pre-matching lottery. Section 2.3 describes the basic model. Section 2.4 shows the dominance of extremely volatile gambles when the surplus function is bi-linear. Section 2.5 highlights how stability fosters risk-taking behaviors. Section 2.6 shows the existence of the efficient equilibrium and the equilibrium properties. Section 2.7 extends the model to a dynamic setting and discusses interplay between occupational choices and marital timing. Section 2.8 concludes.

Related Literature

The paper is linked to the literature that answers why people gamble. Smith (1776) raises and discusses the question. The primary consideration has been the additional social status wealth adds. Modern treatment begins with Friedman and Savage (1948). They argue the utility over money is convex on certain range, and people take gambles if their wealth falls in that range. Their explanation for the convexity is that people gamble with wealth because of social status considerations. Wealth gain raises not only consumption but also one's social status. Ray and Robson (2012) formalizes these arguments in a dynamic setting. Becker et al. (2005) investigate the gambling incentives when social status can be bought and traded in an explicit or implicit market. Complementarity between money and status for utility is key to prompt risk-taking behavior that results in unequal wage distributions. This paper shows complementarity is no longer key in a two-sided hedonic market.

The idea that marriage can generate gambling incentives has been explored before as well. Rubin and Paul (1979) and Robson (1996) both discuss incentives to gamble in order to obtain additional wives. Because of discreteness number of wives a man can have, he would rather take unfair lotteries to obtain enough wealth and resources for an additional wife than to let some of resources to remain idle for inefficient use. However, the papers rely on both assumptions of non-transferable utility and polygamy. The current paper is in a transferable utility, one-to-one matching setting. Roughly speaking, the previous papers have considered how quantity of wives can affect a man's gambling incentive, whereas the current paper considers how quality can induce similar gambling incentive.

Rosen (1997) shows agents' willingness to gamble if having more wages enable one to move to bigger cities with more abundant resources. From the literature that investigates voluntary and wealth redistribution, the paper provides a possible relationship between efficiency and inequality in a matching market. Rosen (1997) and Becker et al. (2005) investigate wealth redistribution as a result of gambling incentives. Bergstrom (1986) studies gambling and occupational choices. Anderson and Smith (2010), although not on gambling incentives, show the convexity of the payoff function when the surplus function is bilinear and supermodular, and some of their key results depend on this feature of the payoff function. This paper generalizes this result and investigates the persistent underlying driving force.

Different from the previous investment-and-matching models in which investments yield deterministic returns, I investigate investments with uncertain returns. The literature on investments with deterministic returns is fairly large (Cole et al., 2001b; Iyigun and Walsh, 2007; Chiappori et al., 2009; Mailath et al., 2012, 2013; Dizdar, 2013). Zhang (2015a) studies investments with uncertain returns but the focus is not on the tradeoff between investments of different volatilities. We can draw some parallels about equilibrium results. The efficient gambles and inequality can be supported in equilibrium, but there may also exist inefficient equilibria. The inefficiency is related to the lack of corresponding agents on the other side of the market, similar to the causes of inefficiency in Cole et al. (2001b) and Dizdar (2013). The equilibrium gambles and the deterministic investments are constrained efficient, however.

2.2 A Motivating Example

Let me first show a simple example in which an agent always embraces uncertainty and prefers a gamble over a deterministic investment that has the same expected return. Consider a two-sided matching market with continuums of men and women distinguished by a one-dimensional matching attribute denoted by x and y, and the attributes are distributed according to mass distributions G_m and G_w. A man of attribute x and a woman of y produce zero individually and positive surplus $s(x, y)$ if matched. Market conditions determine the matching among agents and division of the surplus. Matching π and payoffs u and v are said to be stable if the matching maximizes total surplus $\int s d\pi$ and matched partners split their surpluses, i.e. $u(x)+v(y) = s(x, y)$ if $\pi(x, y) > 0$, and no pair of agents can be strictly better off matching with each other and redividing their surplus than staying with their current partners, i.e. for any x and y,

$$u(x) + v(y) \geq s(x, y).$$

This inequality holds for *any* pair of agents and often holds strictly for any two agents who are not matched to each other.

Now suppose that the surplus function is $s(x, y) = xy$ and that a risk-neutral man born with attribute x has an investment choice before entering the large matching market, while the rest of the agents can only enter the market with their innate attributes. He can either enter the large matching market with attribute x or take a fair gamble that changes his attribute such that with probability p he has high attribute $\overline{x} > x$ and with probability $1-p$ he has low attribute $\underline{x} < x$, but the expected attribute remains x, $x = p\overline{x} + (1 - p)\underline{x}$.

Since there are continuums of agents, the man's choice between x and $p \circ \overline{x} + (1 - p) \circ \underline{x}$ does not affect the equilibrium payoff functions $u(\cdot)$ and $v(\cdot)$. He chooses an investment to maximize his expected payoff. If he does not take the gamble, his payoff is $u(x)$. If he takes

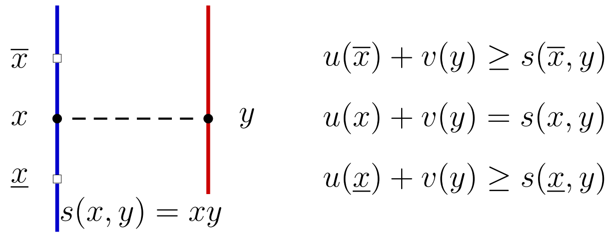

Figure 2.1: Stability conditions in a two-sided matching market.

the gamble, his expected payoff is $pu(\overline{x}) + (1-p)u(\underline{x})$.

In fact, the gamble yields weakly greater expected payoff. If he enters the matching market with attribute x and is stably matched with a woman of attribute y (as shown in Figure 2.1)[1], his payoff is the surplus x and y produces net any payoff y takes,

$$u(x) = xy - v(y).$$

Simply by the stability condition $u(x) + v(y) \geq s(x, y)$ for all x and y, the following two inequalities must hold:

$$u(\overline{x}) \geq \overline{x}y - v(y),$$

$$u(\underline{x}) \geq \underline{x}y - v(y).$$

Figure 2.1 illustrates the matching market and summarizes these implied stability conditions.

The expected payoff of taking the gamble is then bounded by these stability inequalities, in particular,

$$\begin{aligned} pu(\overline{x}) + (1-p)u(\underline{x}) &\geq p[\overline{x}y - v(y)] + (1-p)[\underline{x}y - v(y)] \\ &= p\overline{x}y + (1-p)\underline{x}y - v(y) = xy - v(y) = u(x). \end{aligned}$$

Therefore, taking the fair lottery for x is at least as good as not taking it. Often the

[1] If x is not matched to any woman, $u(x) = 0$, then taking the gamble is trivially weakly better as $u(\cdot)$ is non-negative by individual rationality.

inequalities $u(\overline{x}) \geq \overline{x}y - v(y)$ and $u(\underline{x}) \geq \underline{x}y - v(y)$ hold strictly; for example, when none of the partners of men with attribute $\overline{x}$ is a woman with attribute y in the stable matching ($\pi(x, y) = 0$), $u(\overline{x}) > \overline{x}y - v(y)$. As long as one of the inequalities holds strictly, the fair lottery yields strictly higher expected payoff than the risk-free investment. By continuity, the agent can strictly prefer an unfair lottery even if he is strictly risk averse in the payoffs. The range of such desirable unfair lotteries can be quite large; as shown in the subsequent numerical Example 1, the risk-neutral median man is willing to take a gamble that makes him the greatest with probability 1/4 and the worst with probability 3/4.

Example 1. Mass 1 of men and mass 1 of women are endowed with attributes uniformly distributed on $[0, 1]$. Suppose the surplus function is $s(x, y) = xy$ and single agents produce zero. The stable matching is positive assortative and the unique stable payoff functions are $u(x) = x^2/2$ and $v(y) = y^2/2$. An attribute $x = \frac{1}{2}$ man decides between remaining attribute $\frac{1}{2}$ and taking a gamble that changes his attribute to either the worst $\underline{x} = 0$ or the best $\overline{x} = 1$. If his utility is $U(u) = u^{\frac{1}{2}(1+\rho)}$, he prefers buying an extreme fair lottery $\frac{1}{2} \circ 0 + \frac{1}{2} \circ 1$ to remaining attribute $\frac{1}{2}$ if

$$p \cdot 1^{1+\rho} + (1 - p) \cdot 0^{1+\rho} > (\frac{1}{2})^{1+\rho} \Rightarrow p > 1/2^{1+\rho}.$$

In particular, if the man is risk-neutral ($\rho = 1$), he prefers the extreme lottery to no lottery if $p > 1/4$. If $\rho = 1/2$ so his utility is $U(u) = u^{\frac{3}{4}}$, he prefers the lottery if $p > 0.36$. Of course, gambling is not always preferred when the agent is too risk averse and/or the lottery is too unfair. If the man is risk averse enough, when $\rho < 0$ in this example, he strictly prefers staying attribute $\frac{1}{2}$ to taking any fair lottery. If the available lottery is $p \circ 1 + (1 - p) \circ 0$ where $p < 1/4$, a risk-neutral man strictly prefers not taking it.

In this example, preference for gamble over risk-free option does not rely on particular attribute distributions G_m and G_w. If all agents can choose gambles and their choices result in attribute distributions, each agent nonetheless individually prefers to choose a gamble. In fact, in a general equilibrium model in which every agent can choose among lotteries, if there are several lotteries that have the same mean and are ordered in terms of second order stochastic dominance, when the surplus function is bilinear, the only weakly dominant investment strategy for each agent is the second order stochastically most dominated investment, which we will show in Section 2.4. However, the independence between opposite side heterogeneity and gambling incentives is not generally true. In fact, the attribute distributions crucially determine the gambling incentives.

2.3 The Model

Economy E consists of a continuum of men and women endowed with one-dimensional attributes respectively denoted by $\widehat{x}$ and $\widehat{y}$. Mass functions F_m and F_w describe the distributions of the attributes on compact supports $\widehat{X}$ and $\widehat{Y}$ in $\mathbb{R}_+$. Without loss of generality, assume equal masses of men and women[2].

The economy E operates in two phases, an investment phase followed by a matching phase. In the investment phase, the agents simultaneously choose among fair investment lotteries to stochastically alter their innate attributes. Then in the matching phase, the agents match based on their realized attributes and divide their matching surpluses in a frictionless stable assignment market. Details of the economy precede definition of the equilibrium.

[2] $\widehat{X}$ and $\widehat{Y}$ could be discrete or continuous. If there are unequal masses of men and women, dummy agents can always be added to the shorter side of the market. Dummy agents do not change their attributes and matching with dummy agents does not generate any surplus.

2.3.1 Gambling Phase

In the investment phase, each agent simultaneously and non-cooperatively chooses a fair lottery that alters one's innate attribute. A lottery $P(\cdot|\widehat{x})$ specifies the cumulative distribution function of the realized attribute. Each man $\widehat{x}$'s feasible lottery set $\mathcal{P}(\widehat{x})$ consists of all lotteries that realize attribute on compact support $X = [\underline{x}, \overline{x}]$ and have expected attribute of $\widehat{x}$. The lotteries for women are similarly defined and specified; $Q(\cdot|\widehat{y})$ represents a feasible fair lottery of $\widehat{y}$ and $\mathcal{Q}(\widehat{y})$ the set of feasible lotteries with compact support on $Y \equiv [\underline{y}, \overline{y}]$. Note that $\mathcal{P}(\widehat{x})$ and $\mathcal{Q}(\widehat{y})$ include every mixed strategy as well; for example, mixed strategy $\sigma \circ P_1(\cdot|\widehat{x}) + (1-\sigma) \circ P_2(\cdot|\widehat{x})$ is equivalent to the pure strategy $P(\cdot|\widehat{x}) = \sigma P_1(\cdot|\widehat{x}) + (1-\sigma) P_2(\cdot|\widehat{x})$.

$(P(\cdot|\cdot), Q(\cdot|\cdot))$, or simply (P, Q), represent the population strategies where each $\widehat{x}$ plays $P(\cdot|\widehat{x})$ and each $\widehat{y}$ plays $Q(\cdot|\widehat{y})$. If $\widehat{X}$ and $\widehat{Y}$ are continuous sets, $P(\cdot|\cdot)$ and $Q(\cdot|\cdot)$ are said to be **well-behaved** if they are discontinuous at only a finite number of points and Lipschitz on every interval of continuity points. In particular, well-behaved $P(\cdot|\cdot)$ and $Q(\cdot|\cdot)$ are absolutely continuous on the intervals of continuity points.

2.3.2 Matching Phase

After agents play well-behaved strategies (P, Q), attributes are realized at the beginning of the matching phase. (P, Q) **induce** heterogeneous attribute distributions G_m and G_w where

$$G_m(x) = \int_{\tilde{x} \leq x} P(\tilde{x}|\widehat{x}) dF_m(\widehat{x}), \quad G_w(y) = \int_{\tilde{y} \leq y} Q(\tilde{y}|\widehat{y}) dF_w(\widehat{y}).$$

Let g_m and g_w represent the associated measures. A man of attribute x and a woman of attribute y produce **surplus** $s(x, y)$, and unmatched agents obtain zero surplus. Assume $s : X \times Y \to \mathbb{R}_+$ is twice differentiable and has non-zero cross-partial almost everywhere. Men and women find partners in a **continuum transferable utility assignment market** characterized by measures g_m on X and g_w on Y and the surplus function s, and match and split surpluses in a stable, efficient way.

A stable bargaining outcome of assignment market (g_m, g_w, s) is composed of a surplus-maximizing matching of g_m and g_w, and payoff functions that specify the stable payoffs for all attributes on X and Y. Let $\pi \in \Pi(g_m, g_w)$ be a feasible matching such that measure π on $\text{supp}(g_m) \times \text{supp}(g_w)$ has marginals g_m and g_w. The primal linear program $\mathbb{P}(g_m, g_w, s)$ finds a feasible matching $\pi \in \Pi(g_m, g_w)$ that attains

$$\sup_{\tilde{\pi} \in \Pi(g_m, g_w)} \int s d\tilde{\pi},$$

and the dual linear program $\mathbb{D}(g_m, g_w, s)$ finds payoff functions $u : \text{supp}(g_m) \to \mathbb{R}$ and $v : \text{supp}(g_w) \to \mathbb{R}$ to minimize total payoffs among all stable payoff functions that satisfy

$$u(x) + v(y) \geq s(x, y) \quad \forall (x, y) \in \text{supp}(g_m) \times \text{supp}(g_w).$$

In other words, the solutions $u(\cdot)$ and $v(\cdot)$ attain

$$\inf_{\{(\tilde{u}, \tilde{v}) | \tilde{u}(x) + \tilde{v}(y) \geq s(x,y) \forall (x,y) \in \text{supp}(g_m) \times \text{supp}(g_w)\}} \left(\int \tilde{u} dg_m + \int \tilde{v} dg_w \right).$$

Definition 2. A **stable bargaining outcome** (π, u, v) of assignment game (g_m, g_w, s) is composed of solution $\pi \in \Pi(g_m, g_w)$ to the primal linear program $\mathbb{P}(g_m, g_w, s)$ and solutions $u(\cdot)$ and $v(\cdot)$ to the dual linear program $\mathbb{D}(g_m, g_w, s)$.

Solutions to the primal and dual linear programs exist[3]. The solutions to the two linear programs also achieve the same objective,

$$\max_{\tilde{\pi} \in \Pi(g_m, g_w)} \int s d\tilde{\pi} = \min_{\{(\tilde{u}, \tilde{v}) | \tilde{u}(x) + \tilde{v}(y) \geq s(x,y) \forall (x,y) \in \text{supp}(g_m) \times \text{supp}(g_w)\}} \left(\int \tilde{u} dg_m + \int \tilde{v} dg_w \right).$$

Let $\mathcal{B}(g_m, g_w, s)$ denote the set of stable bargaining outcomes of (g_m, g_w, s), as there is not necessarily a unique stable bargaining outcome.

[3]Interested readers should consult Gretsky et al. (1992, 1999), Chapters 4 and 5 of Villani (2009), and Dizdar (2013) for more detailed formulations and proofs.

In summary, the economy is composed of a simultaneous non-cooperative investment phase in which everyone chooses a lottery and a cooperative assignment phase. Therefore, $(F_m, F_w, \mathcal{P}, \mathcal{Q}, s)$, innate attribute mass distributions, feasible lottery strategies, and surplus function describe an economy E.

2.3.3 Equilibrium

An equilibrium of the economy consists of population strategies and a stable bargaining outcome extended appropriately. Roughly speaking, in an equilibrium, each agent rationally expects his or her matching outcome and respective payoff and chooses the lottery that maximizes his or her expected payoff.

Consider well-behaved strategies (P, Q) and a stable bargaining outcome (π, u, v) of the assignment market (g_m, g_w, s) induced by (P, Q). Suppose a man of innate attribute $\widehat{x}$ deviates to play a feasible strategy $\tilde{P}(\cdot|\widehat{x}) \neq P(\cdot|\widehat{x})$. There are two issues with such unilateral deviation. First, population strategies in which one agent plays $\tilde{P}(\cdot|\widehat{x})$ and every other agent plays according to (P, Q) are no longer well-behaved. Second, $u(\cdot)$ and $v(\cdot)$ by definition are defined respectively on $\text{supp}(g_m)$ and $\text{supp}(g_w)$, so a deviation $\tilde{P}(\cdot|\widehat{x})$ may realize an attribute x not in $\text{supp}(g_m)$ so the man's payoff $u(x)$ when he realizes attribute x is not defined.

We make the following assumptions to solve these issues. When an agent unilaterally deviates from (P, Q), assume the stable bargaining outcome (π, u, v) is unchanged. If the agent's deviation yields an existing attribute, the payoff is well-defined by the payoff functions. On the other hand, if the agent's realized attribute x lies outside $\text{supp}(g_m)$, his payoff is the maximal net surplus he or she can get given the payoff the partner takes; that is, $u(x) \equiv \sup_{y \in \text{supp}(g_w)}[s(x, y) - v(y)]$ if $x \notin \text{supp}(g_m)$. Define extended stable bargaining outcome for any assignment market with matching and payoffs functions extended to support $X \times Y$. Define the payoff $u(x)$ of a non-existent attribute $x \notin \text{supp}(g_m)$ to be

$u(x) = \sup_{y \in \mathrm{supp}(g_w)}[s(x, y) - v(y)]$. Define $v(y)$ for $y \notin \mathrm{supp}(g_w)$ similarly and $\pi(x, y) = 0$ for all $x \notin \mathrm{supp}(g_m)$ or $y \notin \mathrm{supp}(g_w)$.

Definition 3. An **extended stable bargaining outcome** (π, u, v) of an assignment market (g_m, g_w, s) is composed of $\pi : X \times Y \to \mathbb{R}_+$, $u : X \to \mathbb{R}_+$, and $v : Y \to \mathbb{R}_+$ such that (π, u, v) restricted to $\mathrm{supp}(g_m) \times \mathrm{supp}(g_w)$ is a stable bargaining outcome of (g_m, g_w, s), and for $x \in X \backslash \mathrm{supp}(g_m)$ and $y \in Y \backslash \mathrm{supp}(g_w)$, $\pi(x, y) = 0$, $u(x) \equiv \sup_{y \in \mathrm{supp}(g_w)}[s(x, y) - v(y)]$, and $v(y) \equiv \sup_{x \in \mathrm{supp}(g_m)}[s(x, y) - u(x)]$.

We are ready to define the equilibrium of the game, modifying the definitions of the ex-post contracting equilibrium in Cole et al. (2001b) and Dizdar (2013).

Definition 4. Consider $(P^*, Q^*, \pi^*, u^*, v^*)$ composed of well-behaved strategies (P^*, Q^*) and an extended stable bargaining outcome (π^*, u^*, v^*) in economy $E = (F_m, F_w, \mathcal{P}, \mathcal{Q}, s)$. Let G_m^* and G_w^* represent the induced attribution distributions $G_m^*(x) = \int_{\tilde{x} \leq x} P^*(\tilde{x}|\widehat{x}) dF_m(\widehat{x})$ and $G_w^*(y) = \int_{\tilde{y} \leq y} Q^*(\tilde{y}|\widehat{y}) dF_w(\widehat{y})$ and g_m^* and g_w^* the corresponding attribute measures. $(P^*, Q^*, \pi^*, u^*, v^*)$ is a **fulfilled expectations equilibrium** of economy E if

- (π^*, u^*, v^*) is an extended stable bargaining outcome of (g_m^*, g_w^*, s), and
- (P^*, Q^*) are optimal with respect to extended stable bargaining outcome (π^*, u^*, v^*); namely, for all $\widehat{x} \in \widehat{X}$ and $P(\cdot|\widehat{x}) \in \mathcal{P}(\widehat{x})$, $\int u^*(x) dP^*(x|\widehat{x}) \geq \int u^*(x) dP(x|\widehat{x})$, and for all $\widehat{y} \in \widehat{Y}$ and $Q(\cdot|\widehat{y}) \in \mathcal{Q}(\widehat{y})$, $\int v^*(y) dQ^*(y|\widehat{y}) \geq \int v^*(y) dQ(y|\widehat{y})$.

2.4 Dominant Extreme Gambles Under Linear Surplus

In this section, I show that when the surplus function is linear in agent's attribute, it is a weakly dominant strategy to pick extreme lotteries in which agents gamble to realize only either the highest attribute or the lowest attribute. It follows that there exists an equilibrium in which every man and every woman chooses the extreme lottery when the surplus function is bilinear.

Definition 5. An **extreme lottery** $P(\cdot|\widehat{x})$ for an agent of innate attribute $\widehat{x}$ is to realize only the highest or the lowest possible attribute. Given that $\overline{x}$ with probability p and $\underline{x}$ with probability $1-p$ where $p = \frac{\widehat{x}-\underline{x}}{\overline{x}-\underline{x}}$ so that $p\overline{x} + (1-p)\underline{x} = \widehat{x}$.

Proposition 6. *If $s(x, y)$ is linear in x, an attribute $\widehat{x}$ man's only weakly dominant strategy is the extreme lottery.*

Lemma 3. *If $s(x, y)$ is linear in x, every man weakly prefers $\tilde{P}$ to any second-order stochastically dominant lottery P.*

First, let's see that every man prefers his lottery P_1 to the degenerate lottery. Suppose the population strategies are $P(\cdot|\cdot)$ and $Q(\cdot|\cdot)$ resulting in matching market distributions G_m and G_w, if the selection function is β, $u(\cdot)$ and $v(\cdot)$ are the rationally expected payoffs; regardless of the strategies and the selection function, payoffs satisfy the stability conditions. The expected utility of taking the degenerate lottery $P_0(\cdot|\widehat{x})$ for an attribute $\widehat{x}$ man is $U(\widehat{x}, P_0) = u(\widehat{x})$, and on the other hand, the expected utility of taking fair lottery $P_1(\cdot|\widehat{x})$ when others are playing $P(\cdot|\cdot)$ and $Q(\cdot|\cdot)$ and selection function is β is $U(\widehat{x}, P_1, P, Q, \beta) \equiv U(\widehat{x}, P_1) = \int u(x)dP_1(x|\widehat{x})$. Suppose $\widehat{x}$ is matched with y in the stable matching π, then

$$u(\widehat{x}) = s(\widehat{x}, y) - v(y).$$

By stability, for any $x \in X$,

$$u(x) \geq s(x, y) - v(y).$$

Therefore,

$$U(\widehat{x}, P_1) = \int u(x)dP_1(x|\widehat{x}) \geq \int [s(x, y) - v(y)]dP_1(x|\widehat{x}) = \int s(x, y)dP_1(x|\widehat{x}) - v(y).$$

By linearity of the surplus function in x, $\int s(x,y)dP_1(x|\widehat{x}) = s(\int xdP_1(x|\widehat{x}), y)$. Because by definition $P_1(x|\widehat{x})$ is a fair lottery, $\int xdP_1(x|\widehat{x}) = \widehat{x}$. Hence $U(\widehat{x}, P_1) \geq U(\widehat{x}, P_0)$.

In general, take any random variable $\epsilon \sim \Omega$ such that $\mathbb{E}_\Omega(\epsilon) = 0$. Given any payoff function $u(\cdot)$ that satisfies the stability conditions, if $s(x,y)$ is linear in x, $\int u(\widehat{x}+\epsilon)d\Omega(\epsilon) \geq u(\widehat{x})$. $P_{l+1}(\cdot|\widehat{x})$ is a mean-preserving spread of $P_l(\cdot|\widehat{x})$ if and only if there exists a random variable z_l such that $\mathbb{E}[z_l|x] = 0$, $x_{l+1} = x_l + z_l$.

Therefore, just as the lotteries are second order stochastically ranked, they are ranked in terms of expected utilities as well: the more second order stochastically dominated a lottery is, the higher the expected utility, regardless of other players' lottery choices. Therefore, the most second order stochastically dominated lottery is a weakly dominant strategy for each man and woman.

Naturally, there exists an equilibrium in which the men and women all play the dominant strategy to pick the most volatile investment.

Proposition 7. *Suppose the surplus function $s(x,y)$ is bilinear. There exists an ex-post contracting equilibrium in which every agent takes the extreme gamble.*

It is not necessarily the unique equilibrium. Section 2.6 presents an example with multiple equilibria. Nonetheless, the equilibrium in which every agent is voluntarily exposed to maximal uncertainty exists and in fact maximizes social surplus. When both sides of the agent are heterogeneous enough, then taking the most extreme lottery is everyone's unique strictly dominant strategy, so the equilibrium is unique. This example resonates some of the ideas in Cole et al. (2001b), which I will elaborate in Section 2.6.

Proposition 8 (Uniqueness). *Suppose the surplus function is bilinear. If distributions F_m and F_w are differentiable and strictly increasing on full supports X and Y, and all the non-degenerate lotteries $P_l(\cdot|\widehat{x})$ and $Q_n(\cdot|\widehat{y})$ are differentiable and strictly increasing on full supports X and Y, then the population strategies in which each man chooses lottery $P_L(\cdot|\widehat{x})$*

and each woman chooses lottery $Q_N(\cdot|\widehat{y})$ are the unique equilibrium strategies.

From the derivation above, even for a general surplus function, the agent might 1) strictly prefer a nonempty set of unfair lotteries in the neighborhood of the fair lottery, 2) strictly prefer the fair lottery even if he is strictly risk averse, and 3) strictly prefer an unfair lottery even if he is strictly risk averse.

Note that bi-linearity is not equivalent to supermodularity of the surplus function. Although $s(x, y) = xy$ as used in the example in Introduction is bilinear and strictly supermodular, $s(x, y) = x + y - xy$ for $x, y \in [0, 1]$ is bilinear and strictly submodular. Fundamentally similar to the results in this section, Anderson and Smith (2010) show convexity of the stable payoff function when surplus function is bilinear. I will show a force derived from stability and efficiency that persistently encourages pre-matching risk-taking. Also note that if the surplus function is instead strictly biconvex (e.g. $s(x, y) = x^2y^2$), all the propositions go through and the most volatile investments are strictly dominant.

2.5 Stable Reassignment Effect

The motivating single-agent gambling example in Section 2.2 and the results on extreme gambles as unique dominant strategies under linear surplus in the previous section suggest that the stable payoff functions are inherently convex and that the stable organization of the assignment market itself possibly contributes to such payoff convexity. I will show below an inherent stable rematching effect that always generates convexity in the stable payoff functions and thus induces pre-matching gambling. Notably, this gamble-inducing effect arises solely as a byproduct of the stability conditions and exists independently of the surplus function. The independence of the effect's existence to the shape of the surplus function particularly dismisses the probable speculation that complementarity plays the essential role in encouraging gambles in the previous results.

In essence, the stable assignment, in particular, the rematching based on realized attribute, provides an implicit but persistent benefit to gambling. In any stable bargaining outcome, simply by the stability conditions, a man cannot do strictly better by matching with a partner different from the one that he is assigned to and paying the new partner at least her competitive payoff. In other words, in any stable bargaining outcome, any man (or any woman) is assigned to the partner that maximizes his marital payoff. Consider the scenarios between gambling and no gambling. A man who does not gamble will end up with an assigned partner. When he gambles, he will be very likely assigned a different partner based on the realized attribute. This reassigned partner enables him a higher stable payoff than he would get otherwise with the original partner if he does not gamble. This rematching effect always brings an extra benefit to the agent who gambles. The rematching based on realized attribute adds a benefit to the agent who realizes a higher attribute and an insurance if the realized attribute is low.

Let me now formally demonstrate the gamble-inducing stable rematching effect. Consider an assignment market (g_m, g_w, s) induced by the population strategies (P, Q) and an extended stable bargaining outcome (π, u, v) of the market. Take any man $x \in X$. Recall the stability conditions the stable matching and payoffs satisfy. For any woman $y \in \text{supp}(g_w)$,

$$u(x) \geq s(x, y) - v(y).$$

If there is positive measure of matches between x and y, i.e. $\pi(x, y) > 0$, then

$$u(x) = s(x, y) - v(y).$$

Consider $s(x, y) - v(y)$. It represents the payoff x would get if he marries y and pays y her competitive market value $v(y)$. In the stable assignment, x is matched with y only if

$$s(x, y) - v(y) \geq s(x, y') - v(y') \quad \forall y' \in \text{supp}(g_w).$$

In other words, x marries y that gives the highest possible payoff, so

$$u(x) = \sup_{y \in \text{supp}(g_w)} [s(x, y) - v(y)].$$

Let $y(x) \in \text{supp}(g_w)$ denote a woman attribute that satisfies

$$s(x, y(x)) - v(y(x)) = \sup_{y \in \text{supp}(g_w)} [s(x, y) - v(y)],$$

and let $Y(x)$ denote the set of women whose match with x will yield the maximal payoff $\sup_{y \in \text{supp}(g_w)} [s(x, y) - v(y)]$ for x. Therefore in an extended stable bargaining outcome, $u(x) = s(x, y(x)) - v(y(x))$ for all $x \in X$, not only $x \in \text{supp}(g_w)$. In particular, if an agent has innate attribute $\widehat{x}$ and takes a gamble and realizes a different attribute x, his optimally assigned partners in the market probably differ, and the payoff x gets by matching with any of his optimal partners $y(x)$ weakly exceeds the payoff x gets by matching $\widehat{x}$'s partner $y(\widehat{x})$.

$$u(x) = s(x, y(x)) - v(y(x)) \geq s(x, y(\widehat{x})) - v(y(\widehat{x})). \tag{2.1}$$

If $Y(\widehat{x}) \cap Y(x) = \emptyset$, x has for sure a different assigned partner as $\widehat{x}$, and the payoff benefit is strictly positive.

Now, compare a feasible fair lottery $P(\cdot|\widehat{x})$ for an innate attribute $\widehat{x}$ man to his degenerate lottery $p(x|\widehat{x}) = 1_{x=\widehat{x}}$. The difference between the expected payoffs from the two decisions is

$$\mathbb{E}_{P(\cdot|\widehat{x})}[u(x)] - u(\widehat{x}) = \mathbb{E}_{P(\cdot|\widehat{x})}[s(x, y(x)) - v(y(x))] - [s(\widehat{x}, y(\widehat{x})) - v(y(\widehat{x}))].$$

Subtract and add the same term $\mathbb{E}_{P(\cdot|\widehat{x})}[s(x, y(\widehat{x}) - v(y(\widehat{x}))]$, the hypothetical expected payoff a man $\widehat{x}$ would receive for gambling $P(\cdot|\widehat{x})$, producing with the partner of attribute $y(\widehat{x})$, and transferring $v(y(\widehat{x}))$ to her. The expected payoff difference is then rewritten as

$$\begin{aligned} \mathbb{E}_{P(\cdot|\widehat{x})}[u(x)] - u(\widehat{x}) \quad &= \quad \mathbb{E}_{P(\cdot|\widehat{x})}[s(x, y(x)) - v(y(x))] - \mathbb{E}_{P(\cdot|\widehat{x})}[s(x, y(\widehat{x}) - v(y(\widehat{x}))] \\ &\quad + \mathbb{E}_{P(\cdot|\widehat{x})}[s(x, y(\widehat{x})) - v(y(\widehat{x}))] - [s(\widehat{x}, y(\widehat{x})) - v(y(\widehat{x}))]. \end{aligned}$$

Combine the first two terms and the last two terms respectively, the expected payoff difference is expressed as the sum of two effects,

$$\underbrace{\mathbb{E}_{P(\cdot|\widehat{x})}\left\{[s(x,y(\widehat{x}))-v(y(\widehat{x}))]-[s(\widehat{x},y(\widehat{x}))-v(y(\widehat{x})]\right\}}_{\text{surplus contribution effect}}$$
$$+$$
$$\underbrace{\mathbb{E}_{P(\cdot|\widehat{x})}\left\{[s(x,y(x))-v(y(x))]-[s(x,y(\widehat{x}))-v(y(\widehat{x}))]\right\}}_{\text{stable rematching effect}}$$

The first term represents the difference between two expected payoffs, 1) the expected payoff $\widehat{x}$ gets by taking the gamble $P(\cdot|\widehat{x})$ but matching with the partner $y(\widehat{x})$ he would have without gambling, and 2) the (expected) payoff of $\widehat{x}$ for not gambling. Regardless of the realization of the attribute, the wife gets her share of payoff $v(y(\widehat{x}))$. Therefore, the sign of the second term amounts to the convexity of the surplus function with respect to the agent's own attribute given a fixed partner $y(\widehat{x})$,

$$\mathbb{E}_{P(\cdot|\widehat{x})}\left\{[s(x,y(\widehat{x}))-v(y(\widehat{x}))]-[s(\widehat{x},y(\widehat{x}))-v(y(\widehat{x})]\right\}=\mathbb{E}_{P(\cdot|\widehat{x})}s(x,y(\widehat{x}))-s(\widehat{x},y(\widehat{x})).$$

If the surplus function is convex in x, in other words, man's marginal surplus contribution increases as his attribute increases, then the effect is positive. If the surplus function is concave, or the marginal surplus decreases as man's attribute increases, then the effect is negative. I call the term the **surplus contribution effect** since its sign depends on the slope of marginal surplus function and the convexity of the surplus function. If the surplus function is linear in x, for example when $s(x,y)=xy$, then this term is always zero and this effect does not affect people's gambling incentives at all. Therefore, the previous results on beneficial gambling must be driven by the second effect.

The second term represents the expected payoff difference from optimal partner rematching based on different gambling realizations, whereas the first term represents the expected payoff difference between gambling and no gambling if the agent is held fixed to his no-gambling partner. For any realized $x \in \text{supp}(p(\cdot|\widehat{x}))$,

$$[s(x, y(x)) - v(y(x))] - [s(x, y(\widehat{x})) - v(y(\widehat{x}))]$$

represents the difference between the maximal payoff x can get by matching with $y(x)$ and the possibly non-optimal payoff x can get by matching with $y(\widehat{x})$. By (2.1) and that $s(x, y(x)) - y(x)$ is the maximal payoff for x, the payoff difference is non-negative for any realized x. Since the payoff difference is non-negative for any realization, the expected payoff difference over all possible realizations is always non-negative. I call the second term the **stable rematching effect** for the payoff gain comes from reassigning the agent to an optimally chosen partner by the stability conditions. This weakly positive benefit gives a persistent reason for agents to gamble.

Note that the non-negative stable rematching effect does not depend on any assumption about the shape of the surplus function, but rather comes solely from the stability conditions that govern the assignment market.

To emphasize that stable assignment induces gambles regardless of the shape of the surplus function, let's take a price-theoretic approach to see why in particular surplus supermodularity assumption can be dispensed with. Consider an assignment market (g_m, g_w, s). Suppose that the mass distributions G_m and G_w are strictly increasing and twice differentiable on full supports X and Y and that s is strictly supermodular and twice differentiable. The stable matching is then positive assortative. Let $Y(x) = \{y : \pi(x, y(x)) > 0\}$ the set of women's attributes that attribute x men are matched with. The assumptions on the distributions and the surplus function dictate that $Y(x)$ is a singleton for all x, denoted by $y(x)$; $y(x) = G_w^{-1}(G_m(x))$ is a strictly increasing and bijective function. Moreover, u and v are differentiable. An attribute x man's payoff is the surplus x and $y(x)$ produce together net the payoff of attribute $y(x)$ woman,

$$u(x) = s(x, y(x)) - v(y(x)).$$

Each man chooses and is paired with the woman that maximizes his payoff, so we have the first order condition

$$s_2(x, y(x)) - v'(y(x)) = 0. \tag{2.2}$$

Let's examine the convexity of the continuous and twice differentiable payoff function u. Differentiate with respect to x,

$$u'(x) = s_1(x, y(x)) + [s_2(x, y(x)) - v'(y(x))]y'(x).$$

so by the FOC (2.2), the second term is zero, and $u'(x)$ is simply the marginal surplus of x given the fixed optimal partner $y(x)$,

$$u'(x) = s_1(x, y(x)),$$

a standard and widely result in the matching literature. Differentiate $u'(x)$ with respect to x, we get the differentiation with respect to the first term and to the second term, respectively,

$$u''(x) = \underbrace{s_{11}(x, y(x))}_{\text{surplus contribution effect}} + \underbrace{s_{12}(x, y(x))y'(x)}_{\text{stable rematching effect}}.$$

The two terms correspond to the two effects described above - the surplus contribution effect and the stable rematching effect. The second term, the stable rematching, is undoubtedly always non-negative. When the surplus function is strictly supermodular $s_{12} > 0$, the stable matching is positive assortative so $y'(x) > 0$, so the effect is positive.

Economically, when the surplus is strictly supermodular, it is straightforward to understand gambling incentives to improve expected equilibrium marginal surplus. With a supermodular surplus function, an agent's own marginal surplus increases in partner's attribute, so the agent has incentive to take a fair gamble to be matched with a better partner and enjoys a higher marginal surplus. The gambling incentives in the one-sided hedonic market in Rosen (1997) and Becker et al. (2005) for example crucially depend on the assumption of complementarity.

Although the gambling incentives in the matching market can be justified in the same way when the surplus is supermodular, that does not imply that gambling incentives crucially depend on the supermodularity assumption. Take the extreme opposite case that the surplus function is submodular, $s_{12} < 0$ (e.g. $s(x, y) = x + y - xy$ for $x, y \in [0, 1]$). An agent's marginal surplus *decreases* as the partner's attribute increases. However, the matching market reassigns the agent a partner based on the realized attribute and a higher realized attribute results in a partner with lower attribute when the surplus is submodular. Consequently, when an agent realizes a higher attribute, his stably assigned partner has a lower attribute than if he does not take the gamble; when an agent realizes a lower attribute, his stably assigned partner has a higher attribute. Isolating to the stable rematching effect, the effective marginal surplus $s_1(x, y(x))$ increases unambiguously with respect to one's attribute change, because $s_{12} < 0$ and $y' < 0$ imply $s_{12}(x, y(x))y'(x) > 0$.

The same stable rematching effect always contributes to the convexity of the payoff function regardless of the underlying surplus function. Suppose the surplus function is strictly supermodular for certain (x, y) and strictly submodular for other certain combination of (x, y). When $s_{12} > 0$, the stable matching is locally positive assortative. On the other hand when $s_{12} < 0$, the stable matching is locally negative assortative. Therefore, $s_{12}(x, y)y'(x) \geq 0$ for all (x, y). Nonetheless, as long as the surplus contribution effect is not significantly negative, the stable rematching effect always encourages gambling behavior.

An important market condition that affects gambling incentive for agents is the degree of diversity on the opposite side of the market. Take the extreme case that all women are born identical and do not gamble. Then $y'(x) = 0$ in any market. The stable rematching effect $s_{12}(x,y)y'(x) = 0$. However, if the other side of the two-sided market is diverse, then gambling becomes more attractive. Mathematically, consider the general expression of the stable rematching effect for a realized x,

$$[s(x, y(x)) - v(y(x))] = \sup_{y \in \text{supp}(g_w)} [s(x,y) - v(y)].$$

As $\text{supp}(g_w)$ expands, any man x's optimal payoff weakly increases. The payoff gain due to rematching increases without affecting the magnitude of the surplus contribution effect. The degree of diversity is important in guaranteeing uniqueness of the equilibrium with socially efficient investments in Cole et al. (2001b). It will also play a crucial role in this model. In particular, I show an example with homogeneous agents on both sides of the market and multiple equilibria.

Therefore, convexity of the stable payoff function in the previous sections hinges on this rematching effect. In the special case of a bilinear surplus function, $s_{11} = s_{22} = 0$, the surplus contribution effect is mute and the stable payoff functions exhibit weak convexity generally and strict convexity when the agents are heterogeneous enough.

The stable rematching effect is similar to a substitution effect. However, the matching market is special in the following sense. Consider a competitively organized market. A vector of goods $\{1, \cdots, N\}$ is available, and a bundle is denoted by $\mathbf{x} = (x_1, \cdots, x_N) \in \mathcal{X}$. Suppose the supply of goods is fixed. Every person is endowed with (possibly different) wage earnings $w_i \in \mathbb{R}_+$ and (possibly different, "well-behaved") utility function $u_i : \mathcal{X} \to \mathbb{R}_+$. They are price takers and denote the vector of prices by $\mathbf{p}$.

The utility a person i derives when he has income w is the maximal utility he can derive by consuming the optimal bundle of goods,

$$u(w) \equiv u(\mathbf{x}) \quad \text{s.t. } \mathbf{p}{\cdot}\mathbf{x}{\leq}\text{w}$$

Let $\mathbf{x}(w)$ denote the optimal bundle of goods when a person has income w.

Consider gambling now. Suppose a person starts with income $\widehat{w}$ and can take a fair gamble on the income. The person purchases the goods after gambling. Then the utility difference between gambling and not gambling is

$$\mathbb{E}[u(w)] - u(\widehat{w}),$$

and express in terms of the explicit utility function,

$$\mathbb{E}[u(w)] - u(\widehat{w}) = \mathbb{E}[u(\mathbf{x}(w))] - u(\mathbf{x}(\widehat{w})).$$

Add and subtract the term $\mathbb{E}\left[u\left(\frac{w}{\widehat{w}}\mathbf{x}(\widehat{w})\right)\right]$, i.e. the expected utility if the agent simply consumes the feasible bundle given any income w by shifting proportionally with respect to his income $\widehat{x}$, the difference becomes

$$\mathbb{E}[u(\mathbf{x}(w))] - \mathbb{E}\left[u\left(\frac{w}{\widehat{w}}\mathbf{x}(\widehat{w})\right)\right] + \mathbb{E}\left[u\left(\frac{w}{\widehat{w}}\mathbf{x}(\widehat{w})\right)\right] - u(\mathbf{x}(\widehat{w})).$$

It combines to have two terms,

$$\mathbb{E}\left[u(\mathbf{x}(w)) - u\left(\frac{w}{\widehat{w}}\mathbf{x}(\widehat{w})\right)\right] + \mathbb{E}\left[u\left(\frac{w}{\widehat{w}}\mathbf{x}(\widehat{w})\right) - u(\mathbf{x}(\widehat{w}))\right].$$

2.6 Equilibrium Properties

Conjecture 1. *For the socially efficient allocation (P^{**}, Q^{**}), there exists an extended stable bargaining outcome function $(\pi^{**}, u^{**}, v^{**})$ such that $(P^{**}, Q^{**}, \pi^{**}, u^{**}, v^{**})$ is a fulfilled expectations equilibrium.*

The effect that drives gambling is to convexify the payoff function, which is endogenous determined in this model. The idea is similar to Friedman and Savage (1948) and the subsequent literature. The key advance here is that the utility function is not assumed to be convex. The convexity arises from a rematching effect unique to the assignment market.

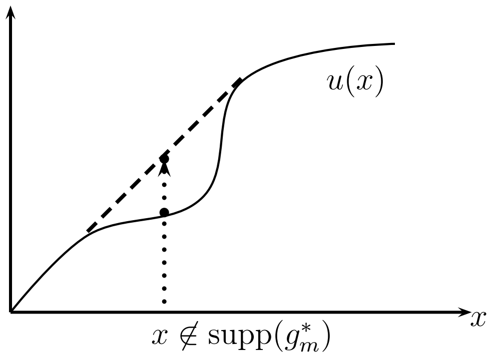

Figure 2.2: Equilibrium payoff functions are weakly concave on equilibrium support.

The efficient equilibrium is not necessarily the unique equilibrium; there may exist inefficient equilibria. In this section, I investigate these inefficient equilibria. An equilibrium is **constrained efficient** if neither side of the agents can collectively change their equilibrium investments to improve total expected surplus. Every equilibrium turns out to be constrained efficient.

Proposition 9. *Every fulfilled expectations equilibrium is constrained efficient.*

If an equilibrium is inefficient, changing one side of investments cannot improve efficiency since it is constrained efficient, but changing both sides of investments can improve. In other words, in every inefficient equilibrium, some coordination failures happen between two sides of the market.

Suppose everyone is born homogeneous and can take a gamble before participating in the matching market. Each of the mass 1 of type $2x_0$ men and mass 1 of type $2y_0$ women can enter the matching market either with the innate type as his or her match type or after taking a lottery P_1 for men and Q_1 for women that with probability $\frac{1}{2}$ a person becomes x_0 and with the other probability $\frac{1}{2}$ the person becomes type $3x_0$. Let the surplus function be $s(x, y) = 2xy$. So the attribute sets are $\widehat{X} = \{x_0, 2x_0, 3x_0\}$ and $\widehat{Y} = \{y_0, 2y_0, 3y_0\}$.

Two quite different rational expectations equilibria arise. In the first equilibrium, no one takes a gamble; in the second equilibrium, everyone takes the gamble.

In the first homogeneous equilibrium, everyone enters the matching market as the innate type without gambling so mass 1 of type $2x_0$ men and mass 1 of type $2y_0$ women are in the matching market. The equilibrium non-negative share payoff functions $u^*(\cdot)$ and $v^*(\cdot)$ satisfy $u^*(x_0) = v^*(y_0) = 0$, $u^*(2x_0) = v^*(2y_0) = 2x_0y_0$, and $u^*(3x_0) = v^*(3y_0) = 4x_0y_0$. Under these equilibrium share payoffs, no one can strictly benefit from taking a gamble.

In the second heterogeneous equilibrium, everyone takes the lottery and the equilibrium share payoff functions are $u^*(x_0) = v^*(y_0) = 0$, $u^*(2x_0) = v^*(2y_0) = 2x_0y_0$, and $u^*(3x_0) = v^*(3y_0) = 4.5x_0y_0$. The matching market is composed of mass 0.5 of type $3x_0$ men, mass 0.5 of type x_0 men, mass 0.5 of type $3y_0$ women, and mass 0.5 of type y_0 women.

The second equilibrium is the efficient equilibrium. The total surplus in the second equilibrium is $\frac{1}{2}(3x_0)(3y_0)+\frac{1}{2}(x_0)(y_0) = 5x_0y_0$ while the total surplus in the first equilibrium is only $(2x_0)(2y_0) = 4x_0y_0$. Therefore, a social planner should encourage the seemingly risky behavior.

A perhaps undesirable effect of the population-wide risky behavior is the ex-post inequality in their match types and payoffs. A remedy to the problem is a progressive tax scheme on the share payoffs. As we see in Section 2.5, the share payoff functions are strictly convex. A progressive tax scheme can flatten out the convexity of share payoff function. As long as there the after-tax share payoffs remain a little convex, incentives to invest and improve still exist. Moreover, a carefully designed progressive tax, besides reducing inequality, can also produce positive revenue and eliminate inefficient equilibrium.

Let me show an example of such tax scheme that can be implemented for the current example. Suppose $\tau_m(1x) = \tau_w(1y) = -1, \tau_m(2x) = \tau_w(2y) = 0, \tau_m(3x) = \tau_w(3y) = \frac{2}{9}$, that is, attribute 1s are given subsidies of 100% and attribute 3s are taxed fraction 2/9 of their earnings. Before the taxes, $u^*(1x) = v^*(1y) = 1, u^*(2x) = v^*(2y) = 3, u^*(3x) = v^*(3y) = 9$. After the taxes, $u^\dagger(1x) = v^\dagger(1y) = 2, u^\dagger(2x) = v^\dagger(2y) = 3, u^\dagger(3x) = v^\dagger(3y) = 7$: the inequality is reduced and the equilibrium lotteries are still supported. Furthermore, a positive tax revenue is generated: $\tau = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (-1) = \frac{1}{2}$. Finally, the previous inefficient equilibrium investments are no longer supported as the risk of a bad outcome is completely insured by the subsidy: $u^\dagger(1x) = v^\dagger(1y) = 4, u^\dagger(2x) = v^\dagger(2y) = 4, u^\dagger(3x) = v^\dagger(3y) = \frac{14}{3}$.

A literature exists on the relationship between economic growth and income inequality. Rosen (1997) focuses on the complementarity between income and location. Becker et al. (2005) focus on market for status and generate endogenous income distributions. Ray and Robson (2012) have an equilibrium growth model with endogenous risk-taking. However, none of the papers considers the gambling incentives in a matching market.

2.7 Occupational Choices with Marital Concerns

We can extend this framework to understand gender differences in risk-taking behavior as well. If we treat the risk-taking behavior as career path choices, we can explain the observation that more men are in risky careers and explain marriage age-personal income relationships. An active literature empirically examines the gender difference in risk preferences and tries to justify the career choice. I provide an alternative explanation to what seem to be a difference in risk preferences - differential fecundity.

Time is discrete and infinite. At the beginning of each period, masses one of men and women are born and are endowed with attributes $\widehat{x}$ and $\widehat{y}$, distributed according to f_m and f_w on $\widehat{X}$ and $\widehat{Y}$. They each live for two periods which we refer to as ages 1 and 2, and make an occupational choice and a marriage timing choice. At the beginning of age 1, they can either take either a safe occupation that has certain returns or a risky occupation that has uncertain returns. The safe occupation S gives an agent lifetime earning of $\widehat{x}$ for sure and the risky occupation R gives the agent an earning of $\widehat{x} + \epsilon$ with probability $\frac{1}{2}$ and $\widehat{x} - \epsilon$ with probability $\frac{1}{2}$. The risky occupation realizes its return at age 2 and cost c_R to choose.

They choose to marry at age 1 or age 2 after choosing occupation. Denote the early marriage choice by E and marriage delay by D. It costs c_D to delay to age 2. The marital surplus $s : X \times Y \to \mathbb{R}_+$ depends on the realized wages x and y; suppose s is continuously differentiable and is weakly concave, $s_{11}, s_{22} \leq 0$. A risky occupational man who enters the marriage market early is of matching type $\frac{1}{2} \circ (\widehat{x} + \epsilon) + \frac{1}{2} \circ (\widehat{x} - \epsilon)$. Suppose every agent is risk-neutral. The expected surplus a man of type $\frac{1}{2} \circ (\widehat{x} + \epsilon) + \frac{1}{2} \circ (\widehat{x} - \epsilon)$ marries a woman of y is $\frac{1}{2} s(\widehat{x} + \epsilon, y) + \frac{1}{2} s(\widehat{x} - \epsilon, y)$.

The marriage market, roughly speaking, is described by (g_m, g_w, s) and returns the stable bargaining outcome.

In the stationary equilibrium, u^* and v^* are fixed. Each agent has four strategies: (S, E), (S, D), (R, E), and (R, D). The expected payoffs for a man $\widehat{x}$ are respectively

- (S, E): $u^*(\widehat{x})$
- (S, D): $u^*(\widehat{x}) - c_{mD}$
- (R, E): $u^*(\frac{1}{2} \circ (\widehat{x} + \epsilon) + \frac{1}{2} \circ (\widehat{x} - \epsilon)) - c_{mR}$
- (R, D): $\frac{1}{2} u^*(\widehat{x} + \epsilon) + \frac{1}{2} u^*(\widehat{x} - \epsilon) - c_{mR} - c_{mD}$

The expected payoffs for a woman $\widehat{y}$ are respectively

- (S, E): $v^*(\widehat{y})$
- (S, D): $v^*(\widehat{y}) - c_{wD}$
- (R, E): $v^*(\frac{1}{2} \circ (\widehat{y} + \epsilon) + \frac{1}{2} \circ (\widehat{y} - \epsilon)) - c_{wR}$
- (R, D): $\frac{1}{2} v^*(\widehat{y} + \epsilon) + \frac{1}{2} v^*(\widehat{y} - \epsilon) - c_{wR} - c_{wD}$

When $c_D > 0$, (S, D) is strictly dominated by (S, E). When $c_R > 0$ and s is concave, (R, E) is strictly dominated by (S, E). The choice is then between (S, E) and (R, D); that is, between choosing the safe occupation and marrying early and choosing the risky occupation and delaying marriage.

A man chooses (R, D) over (S, E) when

$$\frac{1}{2}u^*(\widehat{x}+\epsilon)+\frac{1}{2}u^*(\widehat{x}-\epsilon)-u^*(\widehat{x}) > c_{mR}+c_{mD}.$$

The left hand side is positive because of the stable rematching effect. The gain from delaying marriage is resulted from the different wife one gets. The cost $c_R + c_D$ includes cost of risky occupation and cost of marriage delay.

A woman chooses (R, D) over (S, E) when

$$\frac{1}{2}v^*(\widehat{y}+\epsilon)+\frac{1}{2}v^*(\widehat{y}-\epsilon)-v^*(\widehat{y}) > c_{wR}+c_{wD}.$$

The women have a potentially higher cost when she chooses (R, D).

The simple dynamic model generates a number of stylized facts fairly consistent with our observations. If $c_{wR} + c_{wD} > c_{mR} + c_{mD}$,

- Men are more likely to choose risky occupations than women and marry later.
- Men's wage distributions are more varied than women's.
- Men and women of higher abilities tend to enter risky occupations and marry later.
- Women of higher wages and lower wages tend to marry later.
- If sex ratio is imbalanced, lowest ability men may choose risky activities, for example, criminal activities.

Suppose men and women can choose among a set of lotteries as their wage realizations. However, wages realize over time, and people can only enter the matching market when they realize their wages or know their realizations with certainty. Suppose women's desirability as a marriage partner declines sharply over time, and men's does not decline. Then, men tend to take gambles and women tend to take the non-degenerate lottery or gambles with smaller uncertainty. As the previous sections have shown, gambles are often preferable. Men are without fecundity constraints and therefore tend to choose risky career paths and marry until they are successful in the career pursuits. Women on the other hand have additional waiting costs, so they rather choose career paths that give them steady gains in their incomes and can marry early.

We also can explain the observed personal income-marriage age relationships resulted from the gender differences in career path choices. Around the world, the relationship between age at first marriage and personal income earned in later life has been persistently inverse-U shaped for men: those who marry earlier and later earn significantly less than those who marry around a median age. On the other hand, the correlation between marriage age and personal income for women has been positive until recently in the United States the relationship tends to an inverse-U shape similar to men's.

Suppose men and women of heterogeneous innate types become marriageable in each period and there are costs to invest to choose a set of investment lotteries. For men, those of the lowest abilities do not invest at all and marry as early as possible. Men who choose to invest choose risky lottery options, and those who realize their wage gains marry earlier than those who realize wage gains later or who do not realize the gains at all. Women on the other hand tend to invest safely. Those who do not invest earn low wages and marry early, but those who invest choose career paths that have steady wage gains or that have small variation in wages and realize quicker gains. As a result, the relationship between marriage age and personal income is positive for women if the fecundity constraint is significant. If the fecundity constraint is not as significant as before, for example because of lower demand for quantity of children and a shift to a demand for quality of children, women, more similarly treated as men on the marriage market, will choose uncertain career paths and delay their marriages. Furthermore, the marriage age-personal income relationship tends to be inverse

U as well.

In summary, men and women could have the same risk preferences and still exhibit different risk-taking behavior regarding their career choices, because of their differences in fecundity and the associated social devaluing of older age women.

Zhang (2015a) discusses the influence of fecundity concern on career investment and age at marriage taking the labor market uncertainty as given but such investment lottery options are not available in that model. Adding these lottery options does not change the theoretical predictions. Charles and Luoh (2003) also empirically verify the importance of wage distributions and market opportunities on schooling and the gender differences in those aspects. Preliminary analyses verify the model's predictions.

2.8 Conclusion

I show that stability encourages pre-matching risk-taking behavior. These gambling behaviors are even strictly profitable for risk-averse agents on unfair lotteries. The fundamental force that drives this type of behavior is sorting among the two sides of the market as a result of stability and transferable utilities. This discovery explains some observed risk taking behavior in the market and offers some thought on the inevitability of risk and inequality for economic growth.

It could be interesting to extend these results in various ways. I have shown the possibility of pre-matching gambles in a setting with transferable utility and stability. If agents have ordinal preferences, and the matching among the agents is stable in the sense of Gale and Shapley (1962), we might also investigate agents' risk-taking behaviors in pre-matching investment in these markets. However, utilities and fair lotteries are not straightforwardly defined with respect to ordinal preferences.

Furthermore, it would be interesting to show that the logic goes through in a model with search frictions. Adding search frictions should not change the basic logic that a man still may strictly benefit from fair lotteries and the sorting effect is still strong enough for people to take lotteries. Bi-linearity of surplus function may play a crucial role in the extensions. Burdett and Coles (1997) shows that when the surplus function is bilinear, there is block matching in equilibrium, that is, agents on both sides are segregated into blocks of matching that can be viewed as classes. The matching technology and bargaining process may have different implications for the incentives on pre-matching investments.

2.9 Proofs

Before presenting the proofs, I list some key properties of the extended stable bargaining outcomes to be repeatedly used. Given an assignment market characterized by (g_m, g_w, s), an extended stable bargaining outcome (π^*, u^*, v^*) is composed of the solutions to two linear programs. The primal linear program $\mathbb{P}^*(g_m, g_w, s)$ solves $\sup_{\pi \in \Pi(g_m, g_w)} \int s d\pi$ where $\Pi(g_m, g_w)$ is the set of measures π on $X \times Y$ with marginals g_m and g_w. The dual linear program $\mathbb{D}^*(g_m, g_w, s)$ solves

$$\inf_{\{(u,v)|u(x)+v(y)\geq s(x,y)\forall(x,y)\in(\mathrm{supp}(g_m)\times Y)\cup(X\times\mathrm{supp}(g_w))\}} \left(\int u dg_m + \int v dg_w\right).$$

Solutions to both programs exist and achieve the same objective,

$$\max_{\pi\in\Pi(g_m,g_w)} \int s d\pi = \min_{\{(u,v)|u(x)+v(y)\geq s(x,y)\forall(x,y)\in(\mathrm{supp}(g_m)\times Y)\cup(X\times\mathrm{supp}(g_w))\}} \left(\int u dg_m + \int v dg_w\right).$$

The solutions restricted to $\mathrm{supp}(g_m) \times \mathrm{supp}(g_w)$ constitute the solutions to the linear programs $\mathbb{P}(g_m, g_w, s)$ and $\mathbb{D}(g_m, g_w, s)$, hence a stable bargaining outcome of (g_m, g_w, s).

Furthermore, (π^*, u^*, v^*) satisfies the following properties. For $x \in X$ and $y \in Y$,

$$u^*(x) = \sup_{y\in\mathrm{supp}(g_w)} [s(x, y) - v^*(y)], \quad v^*(x) = \sup_{y\in\mathrm{supp}(g_w)} [s(x, y) - v^*(x)].$$

If $\pi^*(x, y) > 0$, $u^*(x) + v^*(y) = s(x, y)$. Finally, by the constraints of the dual program, for all $(x, y) \in (\text{supp}(g_m) \times Y) \cup (X \times \text{supp}(g_w)$, $u^*(x) + v^*(y) \geq s(x, y)$.

Proof of Proposition 3. Suppose the population strategies are $P(\cdot|\cdot)$ and $Q(\cdot|\cdot)$ resulting in matching market distributions G_m and G_w, if the selection function is β, $u(\cdot)$ and $v(\cdot)$ are the rationally expected payoffs. Take an attribute $\widehat{x} \in X$ man. When all other agents play according to $P(\cdot|\cdot)$ and $Q(\cdot|\cdot)$, and the selection function is β, the expected utility of taking lottery P_{l+1} for $\widehat{x}$ is

$$U(\widehat{x}, P_{l+1}, P, Q, \beta) \equiv U(\widehat{x}, P_{l+1}) = \int u(x_{l+1}) dP_{l+1}(x_{l+1}|\widehat{x}).$$

Since P_{l+1} is a mean-preserving spread of P_l, for $x_{l+1} \sim P_{l+1}(\cdot|\widehat{x})$ and $x_l \sim P_l(\cdot|\widehat{x})$, there exists a random variable z_l with distribution Λ_l and $\mathbb{E}_{\Lambda_l}[z_l|x_l] = 0$ such that $x_{l+1} \overset{d}{=} x_l + z_l$. Then $dP_{l+1}(x_{l+1}|\widehat{x}) = \int_{z_l + x_l \leq x_{l+1}} d\Lambda_l(z_l|x_l) dP_l(x_l|\widehat{x})$. The expected utility of taking lottery P_{l+1} can be rewritten as

$$U(\widehat{x}, P_{l+1}) = \int \left[\int u(x_l + z_l) d\Lambda_l(z_l|x_l) \right] dP_l(x_l|\widehat{x}).$$

For any random variable x such that $\mathbb{E}_\Omega(x) = \widehat{x}$, $\int u(x) d\Omega(x) \geq u(\widehat{x})$. If $\widehat{x}$ is not matched with anyone in the market, $u(\widehat{x}) = 0 \leq u(x)$ for all x. If $\widehat{x}$ is matched with y, $u(\widehat{x}) + v(y) = s(\widehat{x}, y)$. By stability, for any x, $u(x) + v(y) \geq s(x, y)$. Therefore, $\int u(x) d\Omega(x) \geq \int [s(x, y) - v(y)] d\Omega(x) = \int s(x, y) d\Omega(x) - v(y) = s(x, y) - v(y)$ where the last equality follows from linearity of $s(x, y)$ in x.

The inequality holds for any stable payoff function $u(\cdot)$ and any random variable ϵ with zero mean. It holds for z_l with Λ_l so$\int u(x_l + z_l) d\Lambda_l(z_l) \geq u(\widehat{x} + z_l)$. Therefore,

$$U(\widehat{x}, P_{l+1}) = \int \left[\int u(x_l + z_l) dP_l(x_l|\widehat{x}) \right] d\Lambda_l(z_l) \geq \int u(\widehat{x} + z_l) d\Lambda_l(z_l).$$

The expected utility of taking lottery P_l is $U(\widehat{x}, P_l) = \int u(\widehat{x} + z_l) d\Lambda_l(z_l)$, so $U(\widehat{x}, P_{l+1}) \geq U(\widehat{x},$

P_l). Symmetrically, if $s(x, y)$ is linear in y, $U(\widehat{y}, Q_{n+1}) \geq U(\widehat{y}, Q_n)$ for $n \in \{0, \cdots, N-1\}$. $\square$

Proof of Proposition 7. When every man takes lottery P_L and every woman takes lottery Q_N, the induced distributions are $g_m^*(x) = \int p_L(x|\widehat{x})dF_m(\widehat{x})$, and $g_f^*(y) = \int q_N(y|\widehat{y})dF_w(\widehat{y})$. The support of G_m^* and G_f^* can be finite, countable or continuous. Regardless, by Gretsky et al. (1992), stable share payoff functions $u^*(\cdot)$ and $v^*(\cdot)$ exist. By Corollary 6, the strategies are optimal with respect to $\beta^*(g_m^*, g_f^*, s)$. $(P_L(\cdot|\cdot), Q_N(\cdot|\cdot), \beta^*)$ constitutes an equilibrium. $\square$

Proof of Proposition 8. Regardless of the agents' choices, the induced matching market distributions G_m and G_f are strictly increasing and differentiable because of the nice conditions on innate type distributions and lotteries. As a result, the matching function $y(x) = G_w^{-1}(G_m(x))$ is continuous, strictly monotonic, and differentiable. Then any stable share payoff function $u(x)$ is differentiable. In particular, $u''(x) = s_{12}(x, y(x))y'(x) > 0$ is positive. Therefore, given any rationally expected stable payoff functions, an agent strictly prefer to take the lottery that second order stochastically dominates all the others. $\square$

Proof of Proposition 9. Take an equilibrium $(P^*, Q^*, \pi^*, u^*, v^*)$ and equilibrium assignment market distributions (g_m^*, g_w^*) induced by (P^*, Q^*). I prove constrained efficiency of (P^*, Q^*) by contradiction. Suppose to the contrary that there exist population lottery choices $\tilde{P}(\cdot|\cdot)$ such that the surplus-maximizing matching $\tilde{\pi}$ generates surplus $\int sd\tilde{\pi} > \int sd\pi^*$. The distributions induced by $(\tilde{P}, Q^*)$ are $(\tilde{g}_m, g_w^*)$ and let $(\tilde{\pi}, \tilde{u}, \tilde{v})$ be an extended stable bargaining outcome of $(\tilde{g}_m, g_w^*, s)$.

Since the primal and dual linear programs achieve the same objective, $\int sd\tilde{\pi} = \int \tilde{u}d\tilde{g}_m + \int \tilde{v}dg_w^*$ and $\int sd\pi^* = \int u^*dg_m^* + \int v^*dg_w^*$. Therefore,

$$\int sd\tilde{\pi} = \int \tilde{u}d\tilde{g}_m + \int \tilde{v}dg_w^* > \int u^*dg_m^* + \int v^*dg_w^* = \int sd\pi^*. \tag{2.3}$$

Rearrange (2.3) and add the term $\int u^* d\tilde{g}_m$ on both sides of the inequality,

$$\int u^* d\tilde{g}_m - \int u^* dg_m^* > \left[\int u^* d\tilde{g}_m + \int v^* dg_w^*\right] - \left[\int \tilde{u} d\tilde{g}_m + \int \tilde{v} dg_w^*\right]. \tag{2.4}$$

By definition of the dual linear program $\mathbb{D}(\tilde{g}_m, g_w^*, s)$, $(\tilde{u}, \tilde{v})$ restricted to domain $\text{supp}(\tilde{g}_m) \times \text{supp}(g_w^*)$ obtain the minimum of $\int ud\tilde{g}_m + \int vdg_w^*$ among all (u, v) such that $u(x) + v(y) \geq s(x, y)$ for all $(x, y) \in \text{supp}(\tilde{g}_m) \times \text{supp}(g_w^*)$. By the definition of the extended stable bargaining outcome, $(\tilde{u}, \tilde{v})$ defined on $X \times Y$ minimize $\int ud\tilde{g}_m + \int vdg_w^*$ among (u, v) such that $u(x) + v(y) \geq s(x, y)$ for all $(x, y) \in (X \times \text{supp}(g_w^*)) \cup (\text{supp}(\tilde{g}_m) \times Y)$. If $\text{supp}(\tilde{g}_m) \subseteq \text{supp}(g_m^*)$, (u^*, v^*) defined on $X \times Y$ satisfy the condition $u(x) + v(y) \geq s(x, y)$ for all $(x, y) \in (X \times \text{supp}(g_w^*)) \cup (\text{supp}(\tilde{g}_m) \times Y)$ but may not minimize the total surplus, so the RHS of inequality (2.4) is non-negative, and the LHS of the inequality is positive. $\tilde{g}_m$ and g_m^* are respectively induced by $\tilde{P}$ and P^*, so expand $\tilde{g}_m$ and g_m^* on the LHS of (2.4), we get the inequality

$$\int \left[\int u^*(x) d\tilde{P}(x|\widehat{x}) - \int u^*(x) dP^*(x|\widehat{x})\right] dF_m(\widehat{x}) > 0.$$

Therefore, for some $\widehat{x} \in \widehat{X}$, $\int u^*(x) d\tilde{P}(x|\widehat{x}) > \int u^*(x) dP^*(x|\widehat{x})$. However, this conclusion contradicts the equilibrium optimality conditions satisfied by strategies $P^*(\cdot|\cdot)$ and payoff functions $u^*(\cdot)$ that $\forall \widehat{x} \in \widehat{X}$, $\int u^*(x) d\tilde{P}(x|\widehat{x}) \leq \int u^*(x) dP^*(x|\widehat{x})$. We can analogously show Q^* are constrained efficient. $\square$

Chapter 3 The Optimal Sequence of Prices and Auctions

Abstract: This paper highlights the tradeoffs between posting prices and running auctions in a dynamic environment. It also provides a new justification to the use of the buy-it-now option (BIN) on eBay - to post a price before an auction. Consider a seller who must use simple posted prices and reserve price auctions to sell one unit of an indivisible good within a fixed number of periods while buyers with independent private values arrive over time. An auction costs more than a simple posted price and can attract different set of buyers. I characterize the optimal sequence of prices and auctions that maximizes the seller's expected profit. The seller's dynamic programming problem in a finite period setting is non-stationary but remains surprisingly tractable. The optimal mechanism sequence is a sequence of declining prices when the auction cost is sufficiently high, a sequence of auctions with declining reserve prices when the auction cost is sufficiently low, and most interestingly, a sequence of prices followed by a sequence of auctions when the auction cost is intermediate. In particular, auctioning before posting a price is never optimal. The price-auction sequence is optimal under various extensions of the basic setting and resembles a BIN.
Keywords: dynamic mechanism design, reserve price auction, posted price, buy-it-now
JEL: D44

3.1 Introduction

Although a second-price auction with a carefully chosen reserve price can generate the optimal expected revenue when buyers have independent private values (Myerson, 1981), it is usually much more costly and inconvenient to set up than a simple posted price. In comparison with a simple posted price, an auction has operational and mental costs involved with calculating a reserve price, advertising to and aggregating bidders to an auction house, reviewing bids, communicating with buyers, and determining payments. In large competitive markets where the buyers can search and choose among sales of closely substitutable goods, many buyers love posted prices for their certainty of sale price, transparency, immediacy and convenience. As a result, sellers can shy away from running auctions. For example, eBay is an online platform long known for auctions but has evolved to a site mainly of posted prices. In January 2002, more than 90% of the active listings on eBay were auctions, but by late 2012, only around 10% of the active listings were auctions, and the rest were posted prices and hybrid buy-it-now formats (Einav et al., 2013).

In this paper, we explicitly take *auction costs* into account and study how a seller repeatedly chooses between prices and auctions to maximize her expected profit in a dynamic environment. Specifically, a seller must sell one unit of an indivisible good within a fixed T periods and in each period, she either runs a reserve price auction incurring a per-period auction cost or posts a price for free. Buyers with independent private values enter the market each period: they can be short-lived or long-lived, short-sighted or forward-looking. What sequence of prices and auctions should the seller choose to maximize her expected profit? It seems to be daunting to solve for the optimal sequence of prices and auctions, as there are 2^T combinations of them, not to mention the determination of optimal prices and reserve prices in each period. Furthermore, the dynamic programming problem the seller faces in each period changes as the time elapses.

However, it turns out that the non-stationary dynamic programming problem is surprisingly tractable, and the optimal sequence of prices and auctions takes a surprisingly simple form: a sequence of posted prices followed by a sequence of auctions. In the static setting, Myerson (1981) has shown that a reserve price auction with the optimal reserve price determined by the virtual utility curve generates the highest expected revenue; therefore, the seller runs the optimal auction if the auction cost is low, and posts a fixed price otherwise. In the dynamic setting, the seller uses the prices-auctions sequence. Prices are adjusted downward and then auctions are run each period after a period, with the reserve prices declining to the static optimal reserve price in the last period. When the auction cost is sufficiently low, the dynamic pricing phase does not exist: a sequence of auctions is optimal. When the auction cost is sufficiently high, the sequential auction phase does not exist: a sequence of declining prices is optimal. When the auction cost is within most realistic range, as will be numerically shown, the mixed prices-auctions sequence is optimal, and it resembles the buy-it-now option on eBay. No other complicated combination of prices and auctions is optimal. Most notably, it is never optimal for a seller to run auctions then post prices.

The optimality of this simple prices-auctions sequence relies on an implicit endogenous opportunity cost associated with auctions only in the dynamic setting. In the static setting, the seller only faces the straightforward tradeoff between the gain in expected revenue from an auction and the additional cost incurred form an auction. In the dynamic setting, the seller faces an additional cost when he uses an auction. The optimal reserve price is always lower than the optimal posted price, so the probability of a sale using the optimal auction is higher than using the optimal price. In the dynamic setting, the good not sold today is worth the expected revenue it generates in the next period. Therefore, using an auction in a dynamic setting incurs not only an operational cost but also an opportunity cost of selling the good too soon. The operational cost stays constant over time by assumption, but the opportunity cost of selling decreases as the number of remaining periods decreases. Although the seller's dynamic programming problem is non-stationary, the retention value of the good unambiguously diminishes. As a result, running an auction in a later period essentially incurs a lower opportunity cost and thus becomes relatively more attractive.

Having understood this key economic tradeoff between an auction and a posted price, we can easily see that the resulting optimal sequence of mechanisms persists in more general settings when the seller may randomly exit, the buyers enter stochastically and have sequential outside options.

When the horizon is infinite, there is no exogenous deadline to sell the good but a cost of delay that incentivizes early sale, the problem the seller faces in each period is identical, and the seller's optimization problem is stationary. As a result of the stationary problem, the seller runs the same mechanism repeatedly in each period. A high cost seller posts a constant price and a low cost seller runs auctions with a constant reserve price. In particular, the optimal reserve price is inversely related to the auction cost: a lower auction cost delays the sale of the good. Moreover, for a fixed operational cost, it is more likely for a more patient seller to post price. The optimal auction can be approximately implemented by an ascending auction with the deadline determined by the time elapsed after the latest bid. In continuous time, a constant price is always optimal.

This paper solves a dynamic sales mechanism design problem, and contributes to two strands of literature. First, the paper is the first to consider intermingled choices between posted prices and costly auctions. Wang (1993) compares the separate uses of costly auctions and posted prices, and highlights the importance of the steepness of the marginal revenue curve. Kultti (1999) considers the competition between the auctions market and the posted prices market. A large computer science literature highlights the near-optimality of simple sequential posted prices compared to auctions (Blumrosen and Holenstein, 2008; Chawla et al., 2010; Yan, 2011; Hartline, 2012). Although this paper takes a different approach and characterizes the exactly optimal mechanism sequence of prices and auctions, the underlying key message is the same: simple mechanisms such as prices can perform almost equally well as complicated mechanisms.

Second, the paper provides a new justification to the use of buy-it-now options. Previous explanations include risk aversion of the buyers (Budish and Takeyama, 2001), impatience of the seller (Mathews, 2004), and seller and buyer competition (Anwar and Zheng, 2015). In the current setting, the price-auction sequence arises naturally and remains robust under many extensions even when buyers and sellers are risk-neutral and patient. The price-auction sequence has been optimal in various dynamic settings (Riley and Zeckhauser, 1983; McAfee and McMillan, 1988; Dilme and Li, 2012; Board and Skrzypacz, 2014), and the current model is the first to succinctly highlight the importance of the non-stationary opportunity cost associated with running an auction in a dynamic setting. Investigations of mechanism choices in the electronic commerce market have also received attention in the empirical literature (Bajari and Hortacsu, 2004; Einav et al., 2013), and the theoretical investigations in this paper can potentially aide future empirical investigations.

The paper is organized as follows. Section 3.2 introduces the benchmark setting and solves the seller's problem in the static setting. Section 3.3 demonstrates the key insights of the paper with a simple two-period example. Section 3.4 fully solves the seller's problem in the finite horizon. Section 3.5 demonstrates the robustness of the main results under more general settings. Section 3.6 further demonstrates the robustness of the results when buyers are long-lived and forward-looking. Section 3.7 solves and discusses the seller's problem without deadline. Section 3.8 concludes and suggests future extensions.

3.2 Preliminaries and the Seller's Static Problem

In this section, let me first introduce the basic setup, and then characterize the seller's static problem. Many components of the basic setup (e.g. constant discounting, constant fixed buyer arrival, fixed auction cost) seem to be stylized, but will be relaxed significantly in Section 3.5. The main results continue to hold in the extended setting, and the main economic intuitions are adequately demonstrated with the basic setup.

A (woman) seller wants to sell one unit of an indivisible good for which she has zero consumption value. She must sell the good within T periods. T can be one, finitely many, stochastic, or infinite. She discounts each period by the same discount factor $\delta \in [0, 1]$. In each period t, n one-period-lived (man) buyers enter the market. Each buyer has a private value v independently drawn from the identical value distribution F with positive density f on the entire support $[0, 1]$. All the agents are risk-neutral and have quasi-linear preferences in transfers. Except for the private value each buyer is born with, everything else is common knowledge.

Throughout the paper, we maintain the assumption that F has monotone hazard rate. The sole purpose of the assumption is to guarantee that the optimal reserve price and the optimal posted price are uniquely determined, so that we do not have to deal with the irrelevant complication that the optimal mechanism involves ironing.

Assumption 1 (Monotone Hazard Rate Condition)**.** $(1 - F(v))/f(v)$ *is decreasing in* v*.*

At the beginning of each period t, the seller chooses a mechanism m_t, either a reserve price second-price auction A_r or a posted price P_p. A seller running a second-price auction A_r with reserve price r asks each buyer for a sealed bid. At the end of the period the seller assigns the good to the buyer with the highest bid if it is above r, and the winning buyer pays the bigger of the reserve price and the second highest bid. In the second-price auction A_r, it is a dominant strategy for a buyer to bid his value v. A reserve price auction costs $c \geq 0$ to run. On the other hand, it is free to post a price. In a posted price mechanism P_p, the seller posts a fixed price p at the beginning of a period and the buyers with values higher than p have equal chances of receiving the good. It intends to capture a dynamic process in which the buyers sequentially enter the market and each buyer decides immediately whether to purchase or not.

Let $R(m)$ and $\pi(m)$ denote mechanism m's expected revenue and profit in the current period. Any posted price mechanism's expected profit equals its expected revenue, but any auction's expected profit is its expected revenue minus the auction cost c. Let $s(m)$ represent the ex-ante expected probability that the seller sells the good using mechanism m, and let $k(m) = 1 - s(m)$ denote the probability that the good is not sold and is kept by the seller to the next period. The probability that the good is sold from a posted price P_p is $s(P_p) = 1 - F^n(p)$, and the probability that the good is sold from a a reserve price auction A_r is $s(A_r) = 1 - F^n(r)$. Since the probability the good is sold depends only on the posted price or the reserve price, we will write $s(p) = s(P_p)$ and $s(r) = s(A_r)$ for convenience. Similarly, $k(p)$ and $k(r)$ represent the probabilities that the good is kept from using P_p and A_r, respectively.

Let $(m_\tau, \cdots, m_T)$ denote the *mechanism sequence* of the seller who runs mechanism m_t in period t if the good has not been sold by the end of period $t - 1$. The seller's problem is to choose the optimal mechanism sequence $\mathbf{m}^* \equiv (m_1, \cdots, m_T)$ so that the expected profit from running $(m_\tau, \cdots, m_T)$ is maximized for any period τ. Since the buyer arrival process is known and there is no learning by the seller, the seller essentially chooses a sequence of mechanisms at the beginning of the first period, to be executed in each period if the good has not been sold.

The setup intends to capture a seller's problem in a large selling market such as eBay. For example, a person who has bought a lyric opera ticket but could no longer attend the event scheduled in two weeks chooses between posting a fixed price for the ticket and auctioning off the ticket before it loses its value. The item's posting can last for a week (e.g. it stays as a new item for a week on the front page of the website where buyers much more likely search). Potential buyers browse the website and encounter the posting for the item on sale, and decide whether or not to buy the ticket immediately. They have idiosyncratic values for the ticket. Although their individual values are unknown, their aggregate demand curve is known to the seller. Buyers are anonymous to the seller so the seller cannot price discriminate against them, and she is restricted to use either prices or auctions.

The Seller's Static Problem

In the remainder of the section, I solve the seller's one-period problem as a building block for the subsequent multi-period problem. The determination of the optimal reserve price auction is not new and is only restating Myerson (1981). The only addition from the previous literature is the introduction of the posted price's marginal revenue curve that draws parallel to the auction's marginal revenue curve as interpreted by Bulow and Roberts (1989), the virtual utility curve as originally named in Myerson (1981). The introduction of the posted price's marginal revenue curve will especially facilitate the exposition as well as the solution of the seller's problem in the dynamic setting. The advantage will first become apparent in the characterization of the seller's expected profit in the dynamic setting.

Let me first characterize an auction's revenue. The realized revenue of an auction A_r is r if the second highest bid is lower than r, and is v if the second highest bid v is greater than r. Therefore, the expected revenue is

$$R(A_r) = rn[1-F(r)]F^{n-1}(r) + \int_r^1 vn(n-1)[1-F(v)]F^{n-2}(v)f(v)dv.$$

It can be rearranged to be

$$R\left(A_r\right) = \int_r^1 \left[v - \frac{1-F(v)}{f(v)}\right] dF^n\left(v\right).$$

Define

$$\alpha(v) = v - \frac{1-F(v)}{f(v)}.$$

Myerson (1981) call $\alpha(v)$ the virtual utility curve, but we follow Bulow and Roberts (1989) and call $\alpha(v)$ *an auction's marginal revenue curve.* Bulow and Roberts (1989) have named it so because of the following interpretation of the problem. The probability that a buyer buys the good at price v is $q(v) = 1 - F(v)$. An inverse demand curve can be therefore constructed as $v(q) = F^{-1}(1-q)$. The expected revenue from selling quantity q is $\mathrm{R}(q) = q \cdot v(q) =$

$q \cdot F^{-1}(1-q)$, so the marginal revenue is $\mathrm{MR}(q) = F^{-1}(1-q) - q/F'(F^{-1}(1-q))$. Substituting in $v(q)$, $\mathrm{MR}(q(v)) = v - [1-F(v)]/f(v) \equiv \alpha(v)$. A standard auction then becomes a standard monopoly pricing problem. The reserve price auction A_r asks each buyer to submit v and calculate the marginal revenue $\alpha(v)$ accordingly. The good is assigned to the buyer with the highest marginal revenue $\alpha(v)$ if it is positive. Then the expected revenue of any auction A_r is the expected marginal revenue of the highest value buyer when the value exceeds r,

$$R(A_r) = \int_r^1 \alpha(v) dF^n(v).$$

By Myerson (1981), the expected revenue maximizing mechanism among all direct revelation mechanisms is a reserve price auction with the optimal reserve price r^* uniquely determined by $\alpha(r^*) = 0$. Since $\alpha(v)$ is continuous, $\alpha(0) < 0$ and $\alpha(1) = 1$, r^* exists. Since F satisfies the monotone hazard rate condition, $\alpha(v)$ is strictly increasing, and r^* is uniquely determined. The probability the good is not sold is $k(r^*) = F^n(r^*)$.

Posting price p realizes a revenue p if there is a buyer who values it more than p and revenue 0 if no buyer values it more than p. Its expected revenue can be written as $R(P_p) = p[1 - F^n(p)]$. However, we cannot see the connection between a posted price and an auction. As Bulow and Roberts (1989) construct an auction's marginal revenue curve $\alpha(v)$, I construct here a posted price's marginal revenue curve. The auction's marginal revenue curve is constructed for each buyer with value drawn from distribution $F(v)$. The posted price's marginal revenue curve is constructed for the highest value buyer out of the n buyers. Recognize that that the seller generates the same revenue from posting a price to all buyers and from posting the same price to the highest value buyer, as some buyer is willing to pay price p if and only if the highest value buyer is willing to pay price p. The highest value is drawn from the first-order distribution $F^n(v)$. The highest value buyer's inverse demand curve is $v(q) = (F^n)^{-1}(1 - q)$, and the marginal revenue is derived from

$d[q \cdot (F^n)^{-1}(1-q)]/dq$,

$$\rho(v) = v - \frac{1 - F^n(v)}{[F^n(v)]'}.$$

We can succinctly write a posted price P_p's expected revenue as

$$R(P_p) = \int_p^1 \rho(v) dF^n(v).$$

Such representation helps us see the similarities between the two classes of mechanisms. More importantly, the representation facilitates the exposition and eases our understanding in the dynamic setting. The optimal posted price p^* is uniquely determined by $\rho(p^*) = 0$. The optimal price p^* exists because $\rho(v)$ is continuous, $\rho(0) < 0$ and $\rho(1) = 1$. The optimal price p^* is unique because

$$\rho(v) = v - \frac{1 - F^n(v)}{[F^n(v)]'} = v - \frac{F^{n-1}(v) + \cdots + 1}{nF^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)}$$

is strictly increasing in v, as $[F^{n-1}(v) + \cdots + 1]/[nF^{n-1}(v)]$ is strictly decreasing in v, and $[1 - F(v)]/f(v)$ is decreasing in v by Assumption 1. The probability the good is not sold is $k(p^*) = F^n(p^*)$.

Figure 3.1 provides a graphical illustration of the two marginal revenue curves. The optimal reserve price and the optimal posted price equate the marginal revenue to zero. Consequently, the areas under the curves (weighted with respect to $dF^n(v)$) are the expected revenues of the two mechanisms.

Since

$$\rho(v) = v - \frac{F^{n-1}(v) + \cdots + 1}{nF^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)} = \alpha(v) + \left[1 - \frac{F^{n-1}(v) + \cdots + 1}{nF^{n-1}(v)}\right] \frac{1 - F(v)}{f(v)},$$

a posted price's marginal revenue $\rho(v)$ is always smaller than an auction's marginal revenue $\alpha(v)$ except for when $n = 1$ and two curves coincide. Since $\alpha(r^*) = \rho(p^*) = 0$, the optimal posted price is always higher than the optimal reserve price. Clearly from the figure, the

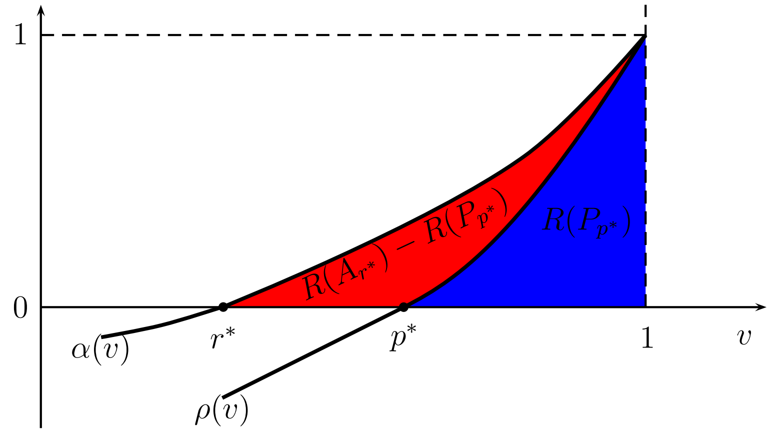

Figure 3.1: Auction's and posted price's marginal revenue curves and expected revenues in the static setting. An auction's marginal revenue curve is $\alpha(v) = v - [1 - F(v)]/f(v)$ and a posted price's marginal revenue curve is $\rho(v) = v - [1 - F^n(v)]/[F^n(v)]'$. The optimal reserve price is $r^* = \alpha^{-1}(0)$ and the optimal posted price is $p^* = \rho^{-1}(0)$. The optimal auction's expected revenue is $R(A_{r^*}) = \int_{r^*}^1 \alpha(v) dF^n(v)$, and the optimal posted price's expected revenue is $R(P_{p^*}) = \int_{p^*}^1 \rho(v) dF^n(v)$. The difference between the optimal revenues can be written as $R(A_{r^*}) - R(P_{p^*}) = \int_0^1 x d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$.

optimal posted price's expected revenue is smaller. Since $\alpha(r^*) = \rho(p^*) = 0$ and $\alpha(1) = \rho(1) = 1$, by changes of variables, the optimal revenue difference can be expressed as

$$R(A_{r^*}) - R(P_{p^*}) = \int_0^1 x d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right]. \tag{3.1}$$

Although the optimal reserve price auction always generates more expected revenue than the optimal posted price, if there is a sufficiently high cost of running the auction, it may be more desirable to post the optimal price. In general, there is a cutoff cost c^* such that the seller with the cost is indifferent between the optimal reserve price auction and the optimal posted price. Running an auction is more appealing for the seller if her cost is lower than the cutoff cost, and posting a price is more appealing if her cost is higher than the cutoff cost. The cutoff cost equals the optimal revenue difference, as the tradeoff in the static setting is simply between the revenue gain from the auction and the cost incurred from the auction. Proposition 10 characterizes the seller's optimal mechanism in the static setting.

Proposition 10. *Suppose $T = 1$. Let r^* and p^* be the unique solutions to $\alpha(r^*) = \rho(p^*) = 0$, and $c^* = R(A_{r^*}) - R(P_{p^*})$. The seller's optimal mechanism is A_{r^*} if $c < c^*$, and is P_{p^*} if $c > c^*$. A $c = c^*$ seller is indifferent between A_{r^*} and P_{p^*}.*

3.3 A Two-Period Example

Let me use a simple two-period example to demonstrate the key insights of the seller's multi-period problem. Suppose that there are two buyers in each of the two periods. Their values are independently drawn from the uniform $[0, 1]$ distribution ($F(v) = v$ and $f(v) = 1$). Purely for simplicity, suppose that the seller does not discount ($\delta = 1$). Her objective is to choose a selling mechanism m_1 in the first period and a selling mechanism m_2 in the second period to maximize her expected profit $\pi(m_1, m_2) = \pi(m_1) + k(m_1)\pi(m_2)$.

We now solve for and characterize the seller's profit-maximizing mechanism sequence (m_1^*, m_2^*). We can solve for the profit-maximizing mechanism choice problem by backward induction. We first solve for the optimal mechanism in the second period. The seller's problem in the last period is essentially a static problem. We thus directly apply the solution of the static problem described in the previous section. The profit of a posted price P_p is $\pi(P_p) = \int_p^1 \rho(v)dv^2$. The optimal price to post in the second period is determined by $\rho(p_2^*) = p_2^* - \frac{1-(p_2^*)^2}{2p_2^*} = 0$, or $p_2^* = \sqrt{3}/3 \approx 0.577$, and the optimal revenue and profit is $R(P_{\sqrt{3}/3}) = \pi(P_{\sqrt{3}/3}) = 2\sqrt{3}/9 \approx 0.385$. The revenue of an auction A_r is $R(A_r) = \int_r^1 \alpha(v)dv^2$. The optimal reserve price is determined by $\alpha(r_2^*) = 2r_2^* - 1 = 0$, so $r_2^* = 1/2$. The optimal revenue is $R(A_{r_2^*}) = \int_{r_2^*}^1 (2v - 1)dv^2 = 5/12 \approx 0.417$. By equation (3.1), let

$$c_2^* \equiv R(A_{r_2^*}) - R(P_{p_2^*}) = \int_0^1 xd\left[\left(\alpha^{-1}(x)\right)^2 - \left(\rho^{-1}(x)\right)^2\right] = \frac{5}{12} - \frac{2\sqrt{3}}{9} \approx 0.032. \tag{3.2}$$

By Proposition 10, if $c < c_2^*$, then the optimal mechanism is $m_2^* = A_{r_2^*} = A_{0.5}$; otherwise, the optimal mechanism is $m_2^* = P_{p^*} = P_{\sqrt{3}/3}$.

Having solved the second period's optimal mechanism, we can solve for the first period's optimal mechanism. Let $\pi_2^* = \pi(m_2^*)$ denote the optimal second period profit. The total expected profit from posting price p_1 in the first period and using m_2^* in the second period is the expected profit posting price p_1 plus the expected profit using m_2^* in case the good is not sold. Since the probability is not sold in the first period is p_1^*, $\pi(P_{p_1}, m_2^*) = R(P_{p_1}) + p_1^2\pi_2^*$. The profit can be written as

$$\pi(P_{p_1}, m_2^*) = \int_{p_1}^{1} \rho(v)dv^2 + \pi_2^* - \int_{p_1}^{1} \pi_2^* dv^2 = \int_{p_1}^{1} [\rho(v) - \pi_2^*]dv^2 + \pi_2^*.$$

The optimal price is thus determined by $\rho(p_1^*) = p_1^* - [1 - (p_1^*)^2]/2p_1^* = \pi_2^*$. The economic interpretation is that the optimal posted price price is set so that the posted price's marginal revenue equates the opportunity cost of selling the good, π_2^*. Solving for the optimal posted price, $p_1^* = (\pi_2^* + \sqrt{\pi_2^{*2} + 3})/3$. The total expected profit from running an auction A_{r_1} in the first period is

$$\pi(A_{r_1}, m_2^*) = R(A_{r_1}) - c + r_1^2\pi_2^* = \int_{r_1}^{1} [\alpha(v) - \pi_2^*]dv^2 + \pi_2^* - c.$$

The optimal reserve price is determined by $\alpha(r_1^*) = r_1^* - (1 - r_1^*) = \pi_2^*$. The economic interpretation is similar as in the optimal posted price determination: the optimal reserve price is set so that the auction's marginal revenue is equated to the opportunity cost of selling the good, π_2^*. Solving for the optimal reserve price, $r_1^* = (\pi_2^* + 1)/2$. Let c_1^* satisfy $\pi(P_{p_1^*}, m_2^*) = \pi(A_{r_1^*}, m_2^*)$; a cost c_1^* seller is indifferent between $P_{p_1^*}$ and $A_{r_1^*}$ in the first period.

$$\begin{aligned} c_1^* &= \left[\int_{r_1}^{1} [\alpha(v) - \pi_2^*]dv^2 + \pi_2^*\right] - \left[\int_{p_1}^{1} [\rho(v) - \pi_2^*]dv^2 + \pi_2^*\right] \\ &= \int_{r_1^*}^{1} [\alpha(v) - \pi_2^*]dv^2 - \int_{p_1^*}^{1} [\rho(v) - \pi_2^*]dv^2. \end{aligned}$$

But remember that $\alpha(r_1^*) = \rho(p_1^*) = \pi_2^*$. With the same change of variable performed for

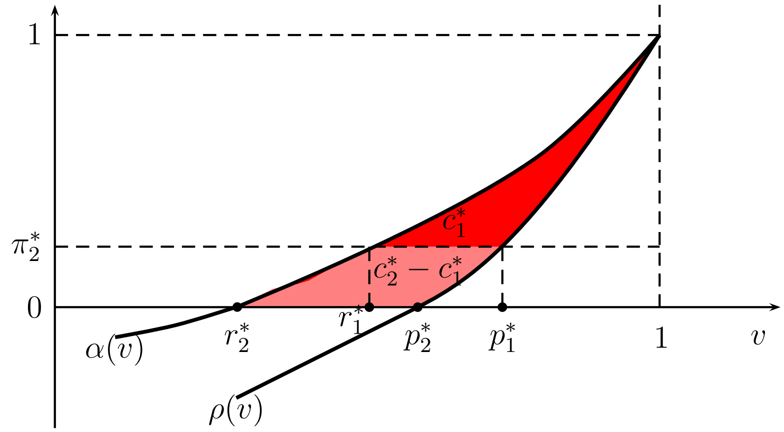

Figure 3.2: Auction's and posted price's marginal revenue curves and expected revenues in the two-period setting. The revenue difference between the optimal auction and the optimal price increases: $c_1^* = \int_{\pi_2^*}^1 [x - \pi_2^*] d[(\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2] < c_2^* = \int_0^1 x d[(\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2]$. The optimal prices decline: $p_1^* = \rho^{-1}(\pi_2^*) > p_2^* = \rho^{-1}(0)$, and the optimal reserve prices decline: $r_1^* = \alpha^{-1}(\pi_2^*) > r_2^* = \alpha^{-1}(0)$.

equation (3.1),

$$c_1^* = \int_{\pi_2^*}^1 (x - \pi_2^*) d \left[\left(\alpha^{-1}(x)\right)^2 - \left(\rho^{-1}(x)\right)^2 \right]. \tag{3.3}$$

Note that π_2^* depends on c. We can solve for $c_1^* \approx 0.0075$. When $c < c_1^*$, auction $A_{r_1^*}$ is used, and when $c \geq c_1^*$, posted price $P_{p_1^*}$ is used.

Let

$$\tilde{c}(\pi) = \int_{\pi}^1 (x - \pi) d \left[\left(\alpha^{-1}(x)\right)^2 - \left(\rho^{-1}(x)\right)^2 \right].$$

Note that $\tilde{c}'(\pi) = -\int_{\pi}^1 d[(\alpha^{-1}(x))^2 - (\rho^{-1}(x))^2] = (\alpha^{-1}(\pi))^2 - (\rho^{-1}(\pi))^2 < 0$. From the characterization of c_1^* in equation (3.3) and c_2^* in equation (3.2), $c_1^* = \tilde{c}(0)$ and $c_2^* = \tilde{c}(\pi_2^*(c_2^*))$. Since $\pi_2^*(c_2^*) > 0$, $c_1^* = \tilde{c}(\pi_2^*(c_1^*)) < \tilde{c}(0) = c_2^*$. Therefore, the optimal mechanism sequence can be characterized in three separate scenarios: $c < c_1^*$, $c_1^* \leq c < c_2^*$, and $c_2 \geq c_2^*$. When $c < c_1^*$, the optimal mechanism sequence is an auction in the first period followed by another in the second period, with lower reserve price in the second period. If $c_1^* \leq c < c_2^*$, the optimal mechanism sequence is a price in the first period followed by an auction in the second period. If $c \geq c_2^*$, the optimal mechanism sequence is a price in the first period

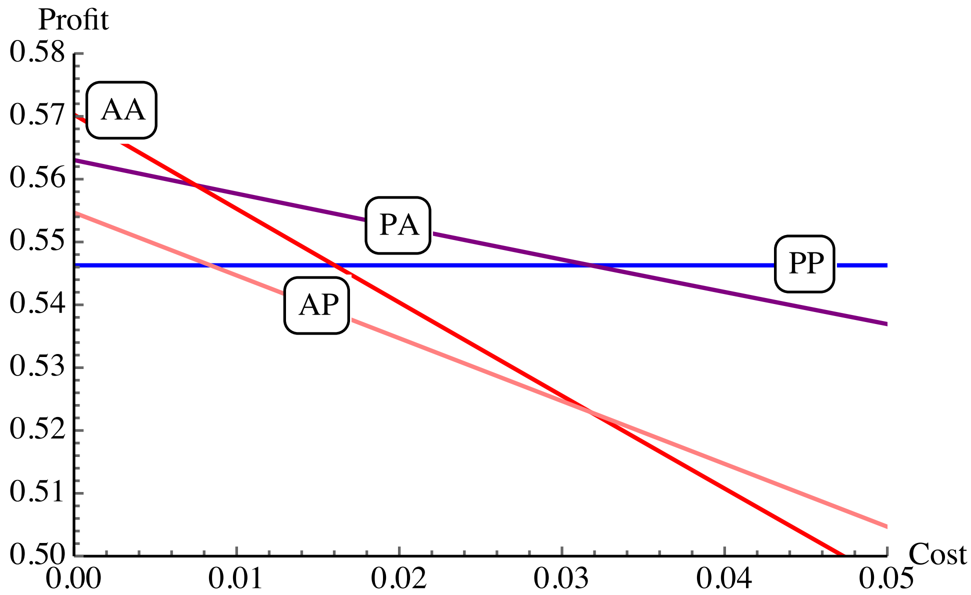

Figure 3.3: The expected revenues of different mechanism sequences as the auction cost varies.

followed by a lower price in the second period. The optimal mechanism's optimal reserve price and optimal posted price are as illustrated in the figure below. Although any of the four combinations is feasible, (A_r, A_r), (A_r, P_p), (P_p, P_p), (P_p, A_r), for any r and p, in this example, the mechanism sequence (A_r, P_p) is never optimal.

The resulting optimal mechanism sequence continues to hold in more general settings: for sufficiently low cost, a sequence of auctions with declining reserve prices is optimal; for sufficiently high cost, a sequence of declining prices is optimal; and for intermediate cost, a sequence of declining prices followed by a sequence of auctions with declining reserve prices is optimal. Any mechanism sequence of auctions followed by prices is never optimal. The optimal sequence of auctions for sufficiently low costs and the optimal sequence of prices for sufficiently high costs are not surprising as they arise almost trivially from the assumption of that auction costs more. However, for intermediate auctioning costs, it is not straightforward to arrive at the conclusion that such a nice sequence of mechanisms is the only optimal mechanism, as any combination of prices and auctions in any order is feasible.

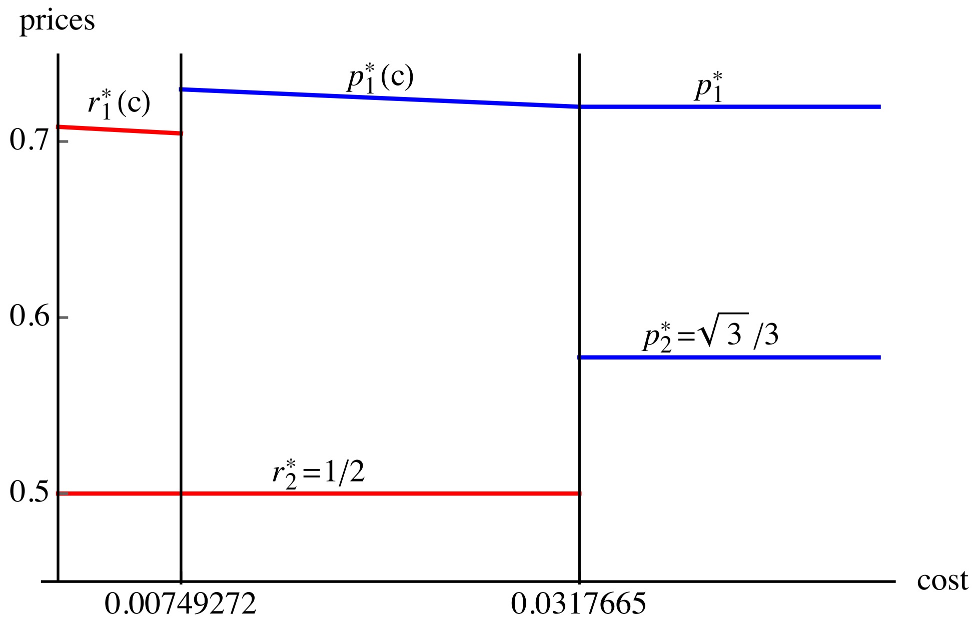

Figure 3.4: The optimal reserve price and the optimal posted price as the auction cost varies.

Although the auction's revenue is always higher, economically, for even economically relevant small cost, the optimal price dominates the optimal auction. In the current example, only when $c < 0.007$, or when the cost is smaller than 1.5% of the expected revenue, auctions are used in both periods. When the cost is between 0.007 and 0.03, a price followed by an auction is optimal. If the costs of the sellers are uniformly drawn between 0 and 0.1, we see roughly that 7% of the sellers use sequential auctions, 25% of the sellers use buy-it-now options, and the majority of the sellers will simply adjust prices. These percentage roughly match the current distribution of selling mechanisms on eBay in magnitude.

Another pattern worth mentioning is that the optimal prices adjust downward over time. Mathematically, we can easily see from the optimal price determination: The optimal reserve prices are determined by $r_2^* = \alpha^{-1}(0)$ and $r_1^* = \alpha^{-1}(\pi_2^*)$, and the optimal posted prices are determined by $p_2^* = \rho^{-1}(0)$ and $p_1^* = \rho^{-1}(\pi_2^*)$. Economically, the optimal prices equate the marginal revenue to the opportunity cost of selling the good. Since the opportunity cost of selling the good decreases from π_2^* in the first period to 0 in the second period, the optimal reserve and posted prices also decrease accordingly.

The rest of the paper is geared towards characterizing the optimal mechanism sequence in more general settings. In particular, we establish in a variety of extensions the optimality of the buy-it-now option with prices preceding auctions, and the sub-optimality of the reverse mechanism sequence - auctions preceding prices.

3.4 Finite Horizon

In this section, we solve the seller's profit maximization problem when T is finite and fixed. For any $t = 1, \cdots, T-1$, the seller's discounted sum of payoffs at period $t < T$ is her expected profit in the current period plus her discounted payoff if the good is not sold in the current period,

$$\pi(m_t, m_{t+1}, \cdots, m_T) = \pi(m_t) + \delta k(m_t) \pi(m_{t+1}, \cdots, m_T).$$

By the Principle of Optimality, we can solve the problem backwards. We have already solved the period T problem, as it has the same solution as the one-period problem. We restate Proposition 10 in the T-period problem.

Lemma 4. *Suppose T is finite. Let $r_T^* = \alpha^{-1}(0)$, $p_T^* = \rho^{-1}(0)$, and $c_T^* = R(A_{r_T^*}) - R(P_{p_T^*})$. The seller's optimal mechanism in the last period, period T, is $m_T^*(c) = A_{r_T^*}$ if $c < c_T^*$, $m_T^*(c) = P_{p_T^*}$ if $c > c_T^*$, and $m_T^*(c) = A_{r_T^*} = P_{p_T^*}$ if $c = c_T^*$.*

The maximized expected profit is the larger of the expected profit of running the optimal auction and of posting the optimal price,

$$\pi_T^*(c) = \max\left\{R(A_{r_T^*}) - c, R(P_{p_T^*})\right\}. \tag{3.4}$$

Given the optimal solution in period T, we can solve for the seller's in period $T-1$. Her problem in period $T-1$ is

$$\max_{m_{T-1}} \quad \pi(m_{T-1}) + \delta k(m_{T-1})\pi_T^*(c)\,.$$

If she chooses an auction A_r in period $T-1$, then her expected profit is

$$\begin{aligned}
\pi(A_r, m_T^*(c)) &= R(A_r) - c + k(r)\delta\pi_T^*(c) \\
&= \int_r^1 \alpha(v)dF^n(v) - c + F^n(r)\delta\pi_T^*(c) \\
&= \delta\pi_T^*(c) + \int_r^1 [\alpha(v) - \delta\pi_T^*(c)]dF^n(v) - c.
\end{aligned}$$

If she chooses a posted price P_p in period $T-1$, then her expected profit is

$$\begin{aligned}
\pi(P_p, m_T^*(c)) &= R(P_p) + k(p)\delta\pi_T^*(c) \\
&= \int_p^1 \rho(v)dF^n(v) + F^n(p)\delta\pi_T^*(c) \\
&= \delta\pi_T^*(c) + \int_p^1 [\rho(v) - \delta\pi_T^*(c)]dF^n(v).
\end{aligned}$$

Her optimal reserve price is $r_{T-1}^*(c) = \alpha^{-1}(\delta\pi_T^*(c))$ and her optimal posted price is $p_{T-1}^*(c) = \rho^{-1}(\delta\pi_T^*(c))$. The seller's optimal profit in period $T-1$ is then

$$\begin{aligned}
\pi_{T-1}^*(c) &= \max\left\{R(A_{r_{T-1}^*(c)}) - c + k(r_{T-1}^*(c))\delta\pi_T^*(c), R(P_{p_{T-1}^*(c)}) + k(p_{T-1}^*(c))\delta\pi_T^*(c)\right\}. \\
&= \delta\pi_T^*(c) \\
&\quad + \max\left\{\int_{r_{T-1}^*(c)}^1 [\alpha(v) - \delta\pi_T^*(c)]dF^n(v) - c, \int_{p_{T-1}^*(c)}^1 [\rho(v) - \delta\pi_T^*(c)]dF^n(v)\right\}.
\end{aligned}$$

There is a cutoff cost c_{T-1}^* such that the seller runs $A_{r_{T-1}^*(c)}$ if her cost is lower than it and runs $P_{p_{T-1}^*(c)}$ otherwise. c_{T-1}^* is determined by

$$\begin{aligned}
c_{T-1}^* &= \int_{r_{T-1}^*(c_{T-1}^*)}^1 [\alpha(v) - \delta\pi_T^*(c_{T-1}^*)]dF^n(v) - \int_{p_{T-1}^*(c_{T-1}^*)}^1 [\rho(v) - \delta\pi_T^*(c)]dF^n(v) \\
&= \int_{\delta\pi_t^*(c_{T-1}^*)}^1 [x - \delta\pi_T^*(c_{T-1}^*)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right].
\end{aligned}$$

The entire optimal mechanism sequence can be solved by iterating this procedure over all periods $t \leq T-1$. The optimal mechanism sequence is summarized in Lemma 5.

Lemma 5. *Suppose T is finite and $t \leq T-1$. Suppose that $\pi^*_{t+1}(c)$ is the optimal expected profit of a cost c seller in period $t+1$. Let $r^*_t(c)$ and $\rho^*_t(c)$ be uniquely determined by*

$$\alpha(r^*_t(c)) = \rho(p^*_t(c)) = \delta\pi^*_{t+1}(c). \tag{3.5}$$

And let

$$c^*_t = \int_{\delta\pi^*_{t+1}(c^*_t)}^{1} [x - \delta\pi^*_{t+1}(c^*_t)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right]. \tag{3.6}$$

A cost c seller's optimal mechanism $m^*_t(c)$ in period $t \leq T-1$ is $A_{r^*_t(c)}$ if $c < c^*_t(c)$, is $P_{p^*_t(c)}$ if $c > c^*_t(c)$, and is $A_{r^*_t(c)} = P_{p^*_t(c)}$ if $c = c^*_t(c)$.

Proof. We need to show that equation (3.6), rearranged as below, has a unique solution c^*_t for each t,

$$\gamma(c^*_t) \equiv c^*_t - \int_{\delta\pi^*_{t+1}(c^*_t)}^{1} [x - \delta\pi^*_{t+1}(c^*_t)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right] = 0. \tag{3.7}$$

It suffices to show that $\gamma(c)$, is continuous, increasing in c, $\gamma(0) < 0$ and $\gamma(1) > 0$. $\gamma(c)$ is differentiable:

$$\begin{aligned}
\gamma'(c) &= 1 + \delta\pi^{*'}_{t+1}(c)\int_{\delta\pi^*_{t+1}(c)}^{1} d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))] \\
&= 1 - \delta\pi^{*'}_{t+1}(c)[F^n(\alpha^{-1}(\delta\pi^*_{t+1}(c))) - F^n(\rho^{-1}(\delta\pi^*_{t+1}(c)))] \\
&= 1 - \delta\pi^{*'}_{t+1}(c)[k(r^*_t) - k(p^*_t)]
\end{aligned}$$

The maximal profit is

$$\pi^*_{t+1}(c) = \max\{R(A_{r^*_{t+1}}) - c + \delta k(r^*_{t+1})\pi^*_{t+2}(c), R(P_{p^*_{t+1}}) + \delta k(p^*_{t+1})\pi^*_{t+2}(c)\}$$

Therefore,

$$\pi^{*'}_{t+1}(c) \geq -1 + \delta k(r^*_{t+1})\pi^{*'}_{t+2}(c),$$

and

$$\pi^{*'}_{t+2}(c) \geq -1 + \delta k(r^*_{t+2})\pi^{*'}_{t+3}(c),$$

and so on for $\pi^{*'}_{\tau}(c)$ through T. Altogether, the inequalities, coupled with the inequalities that $r^*_t > r^*_{t+1}$ for all t, imply that

$$\begin{aligned}
\pi^{*'}_{t+1}(c) &\geq -[1 + \delta k(r^*_{t+1}) + \delta^2 k(r^*_{t+1})k(r^*_{t+2}) + \cdots] \\
&\geq -[1 + \delta k(r^*_{t+1}) + \delta^2 k^2(r^*_{t+1}) + \cdots] \\
&= -\frac{1}{1-\delta k(r^*_{t+1})} \geq -\frac{1}{1-\delta k(r^*_t)}
\end{aligned}$$

Therefore,

$$\gamma'(c) \geq 1 - \frac{\delta k(p^*_t) - \delta k(r^*_t)}{1-\delta k(r^*_t)} > 0.$$

Finally, since $\int_{\delta\pi^*_{t+1}(c)}^1 [x - \delta\pi^*_{t+1}(c)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right]$ is the revenue difference between the optimal auction and the optimal price in a period, it is between 0 and 1 for any c. $\gamma(0) < 0$ and $\gamma(1) > 0$ follow directly. □

The optimal profit in period T is determined by equation (3.4), and the optimal profit in period $t \leq T-1$ is

$$\pi^*_t(c) = \max\left\{R(A_{r^*_t(c)}) - c + k(r^*_t(c))\delta\pi^*_{t+1}(c),\, R(P_{p^*_t(c)}) + k(p^*_t(c))\delta\pi^*_{t+1}(c)\right\}. \tag{3.8}$$

The optimal mechanism sequence is thus completely characterized by the two lemmas and each period's optimal profit function.

Proposition 11. *Suppose T is finite. The optimal mechanism sequence $(m^*_1(c), \cdots, m^*_T(c))$ is characterized by Lemmas 4 and 5, with $\pi^*_t(c)$ defined by equations (3.4) and (3.8), and c^*_t determined by equation (3.6).*

Although Proposition 11 completely solves the seller's problem for any cost c seller, the solution is not very informative. We only know that there is a cutoff cost c^*_t in each period

t such that a seller chooses a reserve price auction if her cost is smaller than c_t^* and posts a price if her cost is bigger than c_t^*. In other words, all we know so far is that in each period if the auction cost is low, use an auction, and if the auction cost is high, post a price.

A little bit more work gives us a much stronger result. We can show that the cutoff costs increase over the periods. In other words, it is more and more likely a seller will run an optimal auction in a later period. Furthermore, we can show that the posted prices and the reserve prices fall over time.

Proposition 12. *Suppose T is finite. The cutoff costs increase over time: $c_1^* < c_2^* < \cdots < c_T^*$. That is, the optimal mechanism sequence is auctions when the auction cost is smaller than c_1^*, is prices-auctions when the auction cost is between c_1^* and c_T^*, and is prices when the auction cost is bigger than c_T^*. Moreover, the optimal prices decrease over time: $r_1^*(c) > r_2^*(c) > \cdots > r_T^*$ and $p_1^*(c) > p_2^*(c) > \cdots > p_T^*$.*

Proof. Equation (3.6) determines the cutoff cost c_t^* in each period. In equation (3.6), the cutoff cost c_t^* is decreasing in $\delta\pi_{t+1}^*(c)$. That is, the lower the optimal discounted sequential profit $\delta\pi_{t+1}^*(c)$ is, the higher the cutoff cost should be. Since the optimal profit decreases over time, $\pi_{t+1}^*(c) > \pi_t^*(c)$. Therefore, $c_t^* < c_{t+1}^*$ for any t. The second part of the proposition follows directly from the determination of optimal posted prices and reserve prices and the facts that $\pi_{t+1}^*(c) > \pi_t^*(c)$ and the marginal revenue curves are increasing. □

The first part of Proposition 12 shows that the cutoff costs are declining over periods so that there is an optimal period of switch from posted prices to auctions. If the auction cost is c, in the optimal mechanism sequence, the seller posts declining prices until period t^* such that $c_{t^*} < c \le c_{t^*+1}$ (define $c_0^* = -\infty$ and $c_{t+1} = \infty$) and then runs auctions from period $t+1$ on. In other words, it is more profitable paying the auction cost later than earlier. In the earlier periods, there are still many more periods left and many opportunities to sell the good. It is not worth paying an auction cost yet because the good has a retention value.

However, as time passes on and the sale opportunities diminish, the auction cost, though constant in absolute terms, is deemed more attractive relative to the risk of not selling the good.

Furthermore, as δ increases, c_t^* decreases for all t. In other words, if the seller is less impatient the seller will post prices in more periods and use auctions in fewer periods. The prices, both optimal fixed prices and reserve prices, are declining over time in equilibrium in finite horizon, as opposed to be constant in infinite horizon. Therefore, fixed deadlines are the main attributors to price declines in this model and in general (Board and Skrzypacz, 2014; Dilme and Li, 2012).

The second part of Proposition 12 shows that the optimal posted prices and the optimal reserve prices are declining over time. The cutoff period t^* can be 0 or T If the cost is too high, the auctions phase does not exist and the seller posts a sequence of declining prices; if the cost is too low, the posted prices phase does not exist and the seller runs a sequence of auctions with declining reserve prices.

As a corollary of the result, it is impossible for any mechanism sequence that has an auction before a posted price to be optimal. In other words, all mechanism sequences consisting of an auction and a posted price following it are always dominated strictly by at least one alternative mechanism sequence - either dynamic pricing, or sequential auctions, or posted price followed by auction.

Proposition 13. *The auction-price mechanism sequence* (A_r, P_p) *is never optimal. Consequently, a mechanism sequence with auctions before prices is never optimal.*

We present a proof that is independent from previous results. It directly constructs the mechanisms that dominate even the expected profit-maximizing auction-price sequence. The marginal revenue curve representations of the expected revenues and the optimal prices are key.

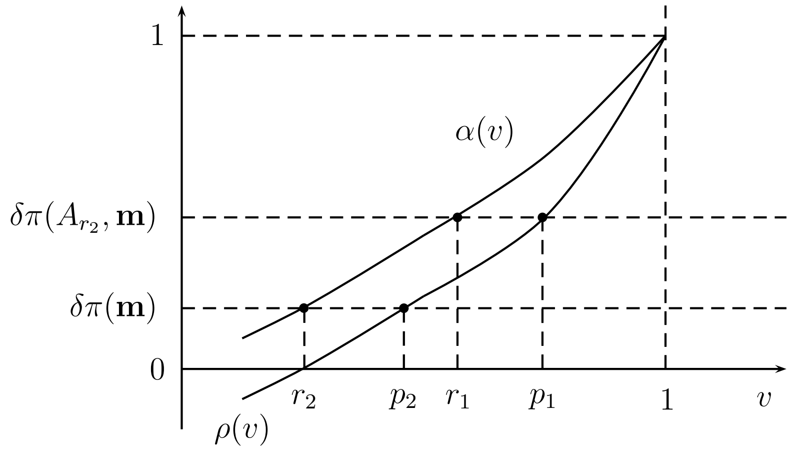

Figure 3.5: Case 1: $r_1 > p_2$. $(A_{r_1}, P_{p_2}, \mathbf{m})$ cannot simultaneously dominate $(A_{r_1}, A_{p_2}, \mathbf{m})$ and $(P_{r_1}, P_{p_2}, \mathbf{m})$.

Proof. Suppose $T \geq 2$. Because the mechanisms chosen before period t do not affect the optimal mechanism sequence after period t, without loss of generality, it suffices to show that the mechanism sequence of an auction in the first period and a posted price in the second period is never optimal. Suppose the seller runs the mechanism sequence $\mathbf{m}$ in periods 3 through T and generates expected profit $\pi(\mathbf{m})$; when there are only two periods, $\pi(\mathbf{m}) = 0$. It suffices to show that the optimal sequence $(A_{r_1}, P_{p_2}, \mathbf{m})$ is dominated by at least one other mechanism sequence not consisting of the auction-price sequence. The optimal r_1 and p_2 are determined respectively by $\rho(p_2) = \delta\pi(\mathbf{m})$ and $\alpha(r_1) = \delta\pi(P_{p_2}, \mathbf{m}) = \delta R(P_{p_2}) + \delta^2 k(p_2)\pi(\mathbf{m})$. If the mechanism sequence is optimal, then $\pi(P_{p_2}, \mathbf{m}) > \pi(\mathbf{m}) > 0$; otherwise, the mechanism sequence $(\mathbf{m}, A_{r_T^*})$ generates strictly higher expected profit. We show separately in the case when $r_1 > p_2$ and in the case when $r_1 \leq p_2$ that the mechanism sequence $(A_{r_1}, P_{p_2}, \mathbf{m})$ is dominated by at least one other mechanism sequence not including auction-price sequence.

Case 1: $r_1 > p_2$.

Suppose $(A_{r_1}, P_{p_2}, \mathbf{m})$ is optimal. The optimality of the mechanism sequence $(A_{r_1}, P_{p_2}, \mathbf{m})$

implies that $(A_{r_1}, P_{p_2}, \mathbf{m})$ dominates both $(A_{r_1}, A_{p_2}, \mathbf{m})$ and $(P_{r_1}, P_{p_2}, \mathbf{m})$,

$$\begin{aligned} \pi(A_{r_1}, P_{p_2}, \mathbf{m}) - \pi(A_{r_1}, A_{p_2}, \mathbf{m}) &\geq 0, \\ \pi(A_{r_1}, P_{p_2}, \mathbf{m}) - \pi(P_{r_1}, P_{p_2}, \mathbf{m}) &\geq 0. \end{aligned}$$

where $\pi(A_{r_1}, P_{p_2}, \mathbf{m})$, $\pi(A_{r_1}, A_{p_2}, \mathbf{m})$ and $\pi(P_{r_1}, P_{p_2}, \mathbf{m})$ can be written out as

$$\begin{aligned} \pi(A_{r_1}, P_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1) R(P_{p_2}) + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}), \\ \pi(A_{r_1}, A_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1)[R(A_{p_2}) - c] + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}), \\ \pi(P_{r_1}, P_{p_2}, \mathbf{m}) &= R(P_{r_1}) + \delta k(r_1) R(P_{p_2}) + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}). \end{aligned}$$

The two inequalities become

$$\begin{aligned} R(P_{p_2}) - [R(A_{p_2}) - c] &\geq 0, \\ [R(A_{r_1}) - c] - R(P_{r_1}) &\geq 0. \end{aligned}$$

Adding the two inequalities up,

$$R(A_{r_1}) - R(P_{r_1}) \geq R(P_{p_2}) - R(A_{p_2}).$$

Since $R(A_r) - R(P_r) = \int_r^1 [\alpha(v) - \rho(v)] \, dF^n(v)$ is strictly decreasing in r, evident in Figure 3.5, when $r_1 > p_2$,

$$R(A_{r_1}) - R(P_{r_1}) < R(P_{p_2}) - R(A_{p_2}),$$

a contradiction with the premise that $(A_{r_1}, P_{p_2}, \mathbf{m})$ is optimal.

Case 2: $r_1 \leq p_2$.

Suppose that $(A_{r_1}, P_{p_2}, \mathbf{m})$ is optimal. It dominates both mechanism sequences $(P_{p_2}, P_{p_2}, \mathbf{m})$ and $(A_{r_1}, A_{r_2}, \mathbf{m})$ where $\alpha(r_2) = \delta\pi(\mathbf{m})$,

$$\pi(A_{r_1}, P_{p_2}, \mathbf{m}) - \pi(P_{p_2}, P_{p_2}, \mathbf{m}) \geq 0,$$

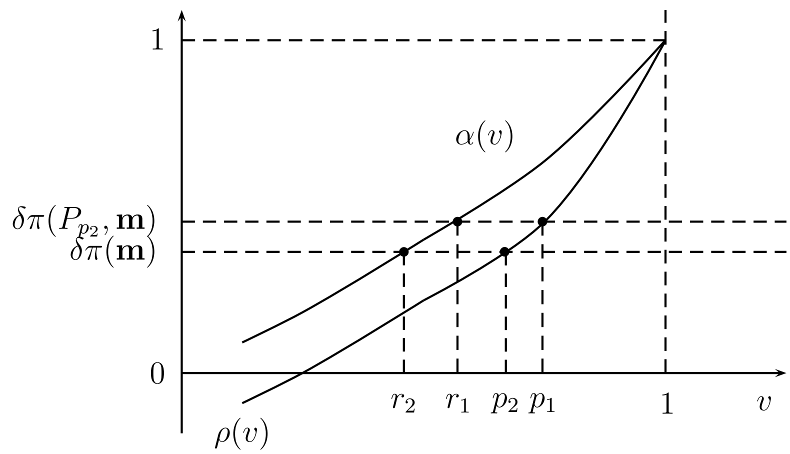

Figure 3.6: Case 2: $r_1 \le p_2$. $(A_{r_1}, P_{p_2}, \mathbf{m})$ cannot simultaneously dominate $(P_{p_2}, P_{p_2}, \mathbf{m})$ and $(A_{r_1}, A_{r_2}, \mathbf{m})$

$$\pi(A_{r_1}, P_{p_2}, \mathbf{m}) - \pi(A_{r_1}, A_{r_2}, \mathbf{m}) \ \ \geq \ \ 0,$$

where

$$\begin{aligned}
\pi(A_{r_1}, P_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1) R(P_{p_2}) + \delta^2 k\left(r_1\right) k\left(p_2\right) \pi(\mathbf{m}), \\
\pi(P_{p_2}, P_{p_2}, \mathbf{m}) &= R(P_{p_2}) + \delta k\left(p_2\right) R(P_{p_2}) + \delta^2 k\left(p_2\right) k\left(p_2\right) \pi(\mathbf{m}) \\
\pi(A_{r_1}, A_{r_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k\left(r_1\right) \left[R(A_{r_2}) - c\right] + \delta^2 k\left(p_1\right) k\left(r_2\right) \pi(\mathbf{m})
\end{aligned}$$

Plugging in the profit functions, we can rewrite the two inequalities as

$$\begin{aligned}
R(A_{r_1}) - c - R(P_{p_2}) - \delta\left[k(p_2) - k(r_1)\right] \pi(R_{p_2}, \mathbf{m}) &\geq 0 \\
R(P_{p_2}) - R(A_{r_2}) + c - \delta\left[k(p_2) - k(r_2)\right] \pi(\mathbf{m}) &\geq 0
\end{aligned}$$

Summing the two inequalities,

$$R(A_{r_1}) - R(A_{r_2}) - \delta\left[k(p_2) - k(r_1)\right] \pi(R_{p_2}, \mathbf{m}) - \delta\left[k(p_2) - k(r_2)\right] \pi(\mathbf{m}) \geq 0.$$

r_1 and r_2 satisfy $\alpha(r_2) = \delta\pi(\mathbf{m})$ and $\alpha(r_1) = \delta\pi(P_{p_2}, \mathbf{m})$. By the optimality of $(A_{r_1}, P_{p_2}, \mathbf{m})$, $\pi(P_{p_2}, \mathbf{m}) > \pi(\mathbf{m})$. As illustrated by Figure 3.1, $r_T^* \le r_2 < r_1$, so $R(A_{r_1}) < R(A_{r_2})$. Coupled with the inequality $r_2 < r_1 \leq p_2$,

$$R(A_{r_1}) - R(A_{r_2}) - \delta\,[k(p_2) - k(r_1)]\,\pi(R_{p_2}, \mathbf{m}) - \delta\,[k(p_2) - k(r_2)]\,\pi(\mathbf{m}) < 0.$$

□

We see that a seller's mechanism choice depends crucially on her cost of running an auction. If the cost is high, she will post prices and adjust prices downward as the deadline to sell approaches. This has been the predominant selling mechanism in many markets. However, with the development of Internet and less friction with information transmission, the cost associated with organizing auctions has been reduced and in particular on sites such as eBay where bidding can be automated and the sellers can set a variety of options to tailor their mechanisms. Nonetheless, some disadvantages associated with auctions, mentioned in the Introduction, cannot be eradicated by technological advances.

When the auction cost is at an intermediate level such that auctions and posted prices are both considerably good mechanisms, mechanism sequences other than posting prices become attractive and are implementable by new eBay mechanisms, in particular, bidding with deadline and buy-it-now options. If the cost is very low possibly because she is an adept and/or fervent auctioneer, she always runs auctions and adjusts her reserve price downwards if no one has bought. When the buyers arrive continuously and randomly in the real world, instead of having a short duration for bidding, the seller can specify the closing time of the ascending auction to be dependent of the time of latest submission bid. If the auction cost is still a significant consideration, we show that the seller will try to sell it by posting prices first before considering auctions. On eBay, this can be very closely implemented by an buy-it-now option by specifying a fixed price that can be immediately sold and a reserve price for minimum bid. Although the time of switch to auction is determined by the buyers, the essential cost-saving strategy of using prices first is captured by buy-it-now options.

3.5 Extensions

In the previous, we show two main results: Proposition 12 and Proposition 13. We extend the benchmark setting and show that both results hold in general. We allow the seller to be increasingly impatient, the buyers to arrive stochastically and have outside options. We also consider the cases when the buyers' markets are separate for auctions and posted prices, when the mechanism designer is procuring a contract from potential contractors rather than selling a good to buyers, and when the seller sequentially sells more than one object. In the next section, we show that the main results continue to hold even when buyers are long-lived and forward-looking.

3.5.1 Perishable Good and Increasing Impatience

If the seller survives the market with probability $\mu < 1$ to the next period, the stochastic deadline may result from the seller's uncertainty when or whether her good will be out of favor or when it may be perished or banned to be sold. It has a similar effect as time discounting δ, as the expected payoff of the future periods becomes $\mu\delta\pi_{T+1}$; by redefining $\delta' = \mu\delta$, the problem is the same as before. Even when the seller survives the market less likely over time, that is, μ_t decreases, the main results are unchanged. Because the assertion that discounted payoff in a period is still greater than that in a later period, $\mu_t\delta\pi_t^*(c) > \mu_{t+1}\delta\pi_{t+1}^*(c)$ for any c, the decline of the cutoff c_t^* over time is unaltered.

3.5.2 Stochastic Buyer Arrival

If in each period, the buyer arrival process is stochastic instead of deterministic, the results do not change qualitatively. Suppose that the probability that the number of buyers is n is $q(n)$, then the probability of selling is $s(p) = 1 - \sum q(n) F^n(p)$. The optimal posted price is then determined by

$$\rho(p_t^*) = p_t^* - \frac{1 - F^n(v)}{[F^n(v)]'} = p_t^* + \frac{s(p_t^*)}{s'(p_t^*)} = \delta\pi_{t+1}^*(c).$$

A posted price's expected revenue is $R\left(P_p\right) = ps\left(p\right)$ and an auction's expected revenue is

$$R\left(A_r\right) = \int_r^1 \alpha\left(v\right) d\left[\sum q\left(n\right) F^n\left(v\right)\right].$$

Since the determination of c_t^* is not changed from equation (3.6), the results from the previous section on the form of mechanism sequence carry over.

3.5.3 Buyers with Outside Options

If the buyers have type-dependent outside options in the latter periods, their willingnesses-to-pay are depressed and the seller solves the problem with respect to the buyers' reservation values instead of their values. As long as the willingness-to-pay function $w(v)$ is a concave transformation of the value, the weighted marginal revenue curve is

$$\tilde{\alpha}(v) = v - \frac{1 - F(v)}{f(v)} w'(v).$$

$\tilde{\alpha}(v)$ is still increasing and the main results continue to hold. For example, the willingness-to-pay function $w(v)$ is concave in the setting of Zhang (2013b).

3.5.4 Separate Markets for Prices and Auctions

Suppose that the auction market and the posted price market are separate: n_A buyers will show up for an auction and n_P buyers will show up for a posted price. The revenue difference between running an optimal auction and an optimal price becomes

$$R(A_{r_t^*}, \mathbf{m}) - R(P_{p_t^*}, \mathbf{m}) = \int_{\delta\pi(\mathbf{m})}^1 [x - \delta\pi(\mathbf{m})] d\left[F^{n_A}(\alpha^{-1}(x)) - F^{n_P}(\rho^{-1}(x))\right] \equiv \Delta(\delta\pi(\mathbf{m})).$$

$\Delta(0) > 0$ and $\Delta(1) = 0$, and

$$\Delta'(\pi) = -\int_{\pi}^{1} d\left[F^{n_A}(\alpha^{-1}(x)) - F^{n_P}(\rho^{-1}(x))\right] = F^{n_A}(\alpha^{-1}(\pi)) - F^{n_P}(\rho^{-1}(\pi)).$$

When $n_A > n_P$, i.e. more buyers are in the auction market than in the price market, the characterization of the optimal sequences remains.

In general, if the buyers arrive stochastically, let $q_A(n)$ denote the probability that n buyers arrive when an auction is run, and $q_P(n)$ the probability that n buyers arrive when a posted price is run. The difference between the optimal revenues is then

$$R(A_{r_t^*}, \mathbf{m}) - R(P_{p_t^*}, \mathbf{m}) = \int_{\delta\pi(\mathbf{m})}^{1} [x - \delta\pi(\mathbf{m})] d\left[\sum q_A(n) F^n(\alpha^{-1}(x)) - \sum q_P(n) F^n(\rho^{-1}(x))\right].$$

We say that an auction faces a higher demand than a posted price if $\sum_{i=1}^{n} q_A(i) \leq \sum_{i=1}^{n} q_P(i)$ for all n. Propositions 12 and 13 continue to hold when an auction faces a higher than a posted price.

On the other hand, when the auction market is smaller, the optimal mechanism sequence is either a sequence of auctions or a sequence of prices. Neither the mixed sequence of prices-auctions nor that of auctions-prices is optimal. In other words, while Proposition 13 continues to hold, Proposition 12 holds with the modification: there is a single cost cutoff such that the seller will use a sequence of auctions if the cost is lower than it, and a sequence of prices otherwise.

3.5.5 Procurement Contracts

Throughout, we have considered the setting in which a monopolist is selling a good. We now turn our attention and consider the problem of a procurement contract in which the monopolist is looking for contracts to complete a project with the lowest cost. We show that the optimal mechanism sequence takes the same form. McAfee and McMillan (1988) consider the procurement problem when the communication cost is high and similar to our argument, find that the principal may search sequentially for contractors if the communication is too costly. A principal who needs a cost of 1 to complete the task, finds a contractor to finish a task for a minimum compensation. Suppose n contractors whose true costs c for the project are independently and identically distributed according to G arrive each period; assume G has increasing hazard rate G/g. The principal's choice is the same as the seller in the previous sections. Her auction's expected profit with a requirement of a maximum bid r is

$$\pi(A_r) = \int_0^r \left[1 - c - \frac{G\,(c)}{g\,(c)}\right] dG^n(c) - c.$$

The probability of sale becomes $s(r) = G^n(r)$. A posted price P_p yields expected revenue $(1-p)\,G^n(p)$. The optimal reserve price and the optimal posted price are determined by

$$r_t^* + G(r_t^*)/g(r_t^*) = p_t^* + G^n(p_t^*)/(G^n(p_t^*)') = \delta\pi_{t+1}^*(c). \tag{3.9}$$

In the static setting, the optimal auction yields more expected revenue than the optimal posted price. In a finite horizon, the counterpart of equation (3.6) is

$$c_t^* = R(A_{r_t^*(c_t^*)}) - R(P_{p_t^*(c_t^*)}) - [k(p_t^*(c_t)) - k(r_t^*(c_t))]\,\delta\pi_{t+1}^*(c_t). \tag{3.10}$$

Therefore, the optimal sequence of mechanisms in the procurement problem depends on the auction cost in the same way as the seller's problem in the benchmark setting.

3.5.6 Multiple Objects

Suppose the monopolist has multiple identical copies of an object for sale, and she must sell each object sequentially. The fundamental problem has not changed at all, so the optimal mechanisms do not change. The only change is in notations to carry around another state variable, the number of objects remaining K. An auction in period t yields expected revenue

$$R_t^K(A_r) = R(A_r) - c + \delta k(A_r)\pi_{t+1}^{K-1} + \delta[1 - k(A_r)]\pi_{t+1}^K.$$

π_t^K is increasing in K and decreasing in t. The same proofs apply in the multiple-goods setting.

3.6 Forward-Looking Buyers

Thus far, we have assumed that the buyers are short-lived. In this section, we consider the much more complicated environment in which the buyers are long-lived and forward-looking. Modify the simplest setting of Section 3.3 with two buyers in each of the two periods. In this section, we assume that the two buyers born in the first period continue to be around in the second period and have a chance to buy the good in the second period if the good has not been sold. Furthermore, the seller and the buyers discount by δ (if the seller does not discount, the optimal mechanism is trivial: wait until the last period to run the static optimal auction or post the static optimal price). We solve one by one the optimal price-price, price-auction, auction-auction, and auction-price sequences. Because buyers are forward-looking, we can no longer solve the seller's problem backwards like we have done when buyers are short-lived. The mechanisms and prices chosen in the latter periods affect buyers' behavior in earlier periods and in turn the seller's profit in the earlier periods. The optimal prices are characterized by a system of equations. Nonetheless, the main result remains: the optimal mechanism sequence is auction-auction when the auction cost is sufficiently low, price-auction when the auction cost is intermediate, and price-price when the auction cost is sufficiently high, and auction-price is never optimal. Figure 3.7 illustrates each mechanism sequence's optimal revenue for different auction costs. For only very small auction cost the auction-auction sequence is optimal.

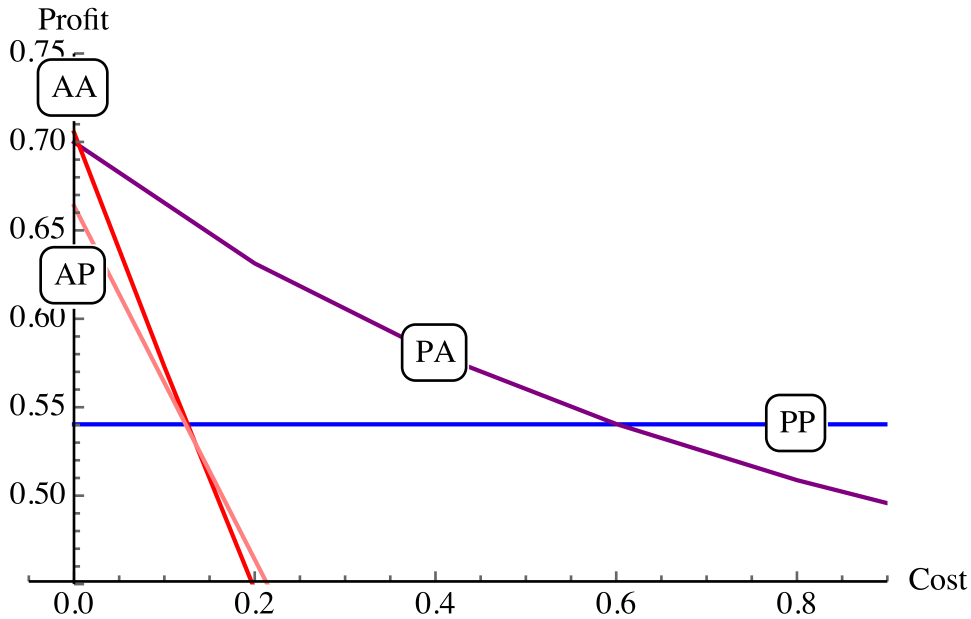

Figure 3.7: The optimal revenues of different mechanism sequences for different auction costs when buyers are forward-looking.

3.6.1 Price-Price Sequence

Suppose the seller posts the price sequence (P_{p_1}, P_{p_2}). We can restrict our attention to the price sequence with declining, because (P_{p_1}, P_{p_2}) with $p_1 \leq p_2$ is never optimal. An upward price path cannot capture the benefits of buyers living longer.

Lemma 6. *(P_{p_1}, P_{p_2}) with $p_1 \leq p_2$ is never optimal.*

Proof. When $p_1 \leq p_2$, buyers who are born with values above p_1 in the first period do not wait until the second period to buy. They will buy in the first period. The expected revenue for the seller is then

$$R(P_{p_1}, P_{p_2}) = \int_{p_1}^{1} \rho(v) dF^n(v) + F^n(v) \int_{p_2}^{1} \rho(v) dF^n(v).$$

The optimal prices are set by $\rho(p_1^*) = R(P_{p_2^*})$ and $\rho(p_2^*) = 0$. But $\rho(\cdot)$ is strictly increasing, $\rho^{-1}(\cdot)$ is strictly decreasing, $p_1^* = \rho^{-1}(R(P_{p_2^*})) > \rho^{-1}(0) = p_2^*$, contradicting the assumption that $p_1^* \leq p_2^*$. ☐

Regardless of the price, there is a cutoff value $\tilde{v} > p_1$ such that first-period buyers with values above $\tilde{v}$ will buy in the first period, and buyers with value below $\tilde{v}$ will wait to buy in the second period. The wait lowers the probability the buyer gets the good, but the wait increases the buyer's utility when he gets the good because of the lower price in the second period. Some buyers with values barely above p_1 are willing to wait.

Lemma 7. *Suppose that the seller runs the mechanism sequence (P_{p_1}, P_{p_2}) with $p_1 > p_2$. There is a cutoff value $\tilde{v}$ such that any buyer with value above $\tilde{v} > p_1$ will buy in the first period, and any buyer with value below $\tilde{v}$ will wait to buy in the second period.*

Proof. Fix a value v buyer. Suppose that his opponent uses a strategy such that he will buy if his value falls into the (compact, possibly disconnected) set $V \subseteq [p_1, 1]$. Let $F(V) \equiv \int_{v \in V} f(v)dv$ be the probability that the opponent buys. Let $V_2 \equiv [p_2, 1] \backslash V$ be the set of the values v such that the value v opponent will buy in the second period instead of the first period. Then a value $v > p_1$ buyer's utility of buying in the first period is

$$u_B(v|P_{p_1}, P_{p_2}) = (v - p_1)[1 - F(V) + \frac{1}{2}F(V)].$$

The utility $u_{NB}(v|P_{p_1}, P_{p_2})$ of not buying in the first period and waiting for the second period is

$$\begin{aligned}(1 - F(V))(v - p_2) \Bigg[&\frac{F^2(p_2)}{1 - F(V)} F^2(p_2) + \frac{1}{2} \frac{F(p_2)}{1 - F(V)} 2F(p_2)(1 - F(p_2)) \\ &+ \frac{1}{2}(F(V) - F(p_2))F^2(p_2) + \frac{1}{3} \frac{F(p_2)}{1 - F(V)} (1 - F(p_2))^2 \\ &+ \frac{1}{3} \frac{F(V) - F(p_2)}{1 - F(V)} 2F(p_2)(1 - F(p_2)) + \frac{1}{4} \frac{F(V) - F(p_2)}{1 - F(V)} (1 - F^2(p_2)) \Bigg].\end{aligned}$$

Both utilities are linear in v, so the two lines $u_B(v)$ and $u_{NB}(v)$ intersect at most at one point. Furthermore, $u_B' > 1 - F(V)$ and $u_{NB} < 1 - F(V)$. Since $u_B(p_1) = 0 < u_{NB}(p_1)$, the intersection is either in $[1, \infty)$, or $(p_1, 1)$. In the first case, $u_{NB}(v) > u_B(v)$ for all $v \in [p_1, 1]$, so no one buys in the first period: we let $\tilde{v} = 1$. In the second case, the intersection $\tilde{v} \in (p_1, 1)$, so a buyer will buy in the first period if his value exceeds $\tilde{v}$. □

The two buyers will play symmetric strategies: he will buy if his value exceeds $\tilde{v}$. A value $\tilde{v}$ buyer is indifferent between buying and not buying in the first period. The utility from buying is

$$u_B(v|P_{p_1}, P_{p_2}) = (\tilde{v} - p_1)[F(\tilde{v}) + \frac{1}{2}(1 - F(\tilde{v}))].$$

The utility $u_{NB}(v|P_{p_1}, P_{p_2})$ of not buying in the first period is

$$\begin{aligned}&(\tilde{v} - p_2)[F^3(p_2) + \frac{1}{2}2F^2(p_2)(1 - F(p_2)) + \frac{1}{2}(F(\tilde{v}) - F(p_2))F^2(p_2)\\&+\frac{1}{3}F(p_2)(1 - F(p_2))^2 + \frac{1}{3}(F(\tilde{v}) - F(p_2))2F(p_2)(1 - F(p_2))\\&+\frac{1}{4}(F(\tilde{v}) - F(p_2))(1 - F^2(p_2))].\end{aligned}$$

$\tilde{v}$ satisfies $u_B(\tilde{v}|P_{p_1}, P_{p_2}) = u_{NB}(\tilde{v}|P_{p_1}, P_{p_2})$. When $F(v) = v$,

$$\begin{aligned}(\tilde{v} - p_1)\frac{1 + \tilde{v}}{2} &= \delta(\tilde{v} - p_2)[p_2^3 + \frac{1}{2}2p_2^2(1 - p_2) + \frac{1}{2}(\tilde{v} - p_2)p_2^2\\&+\frac{1}{3}p_2(1 - p_2)^2 + \frac{1}{3}2(\tilde{v} - p_2)p_2(1 - p_2) + \frac{1}{4}(\tilde{v} - p_2)(1 - p_2^2)] \quad (3.11)\end{aligned}$$

It is possible that the price in period two is so low that no one is willing to buy in the first period. In such a case, let $\tilde{v} = 1$. The seller's expected revenue is

$$R(P_{p_1}, P_{p_2}|p_1 > p_2) = [1 - F^2(\tilde{v})]p_1 + [F^2(\tilde{v}) - F^4(p_2)]p_2. \quad (3.12)$$

When $F(v) = v$, $R(P_{p_1}, P_{p_2}) = (1 - \tilde{v}^2)p_1 + (\tilde{v}^2 - p_2^4)p_2$. In comparison, the revenue when $p_1 < p_2$ is

$$R(P_{p_1}, P_{p_2}|p_1 \leq p_2) = [1 - F^2(p_1)]p_1 + [F^2(p_1) - F^2(p_1)F^2(p_2)]p_2.$$

3.6.2 Price-Auction Sequence

Suppose the seller's mechanism sequence is (P_{p_1}, A_{r_2}) with $p_1 > r_2$. Similarly, we can characterize the buyers' strategies by cutoffs. The utility of buying in the first period is

$$(\tilde{v} - p_1)[F(\tilde{v}) + \frac{1 - F(\tilde{v})}{2}].$$

The utility not buying in the first period is

$$\begin{aligned}
& F^3(\tilde{v}) \left[\tilde{v} - \int_0^{\tilde{v}} \max\{r_2, v\} dF^3(v)/F^3(\tilde{v}) \right] \\
= \; & F^3(\tilde{v}) \left[\tilde{v} - \int_{r_2}^{\tilde{v}} v dF^3(v)/F^3(\tilde{v}) - \int_0^{r_2} r_2 dF^3(v)/F^3(\tilde{v}) \right] \\
= \; & F^3(\tilde{v}) \left[\tilde{v} - (\tilde{v} - \int_0^{r_2} r_2 dF^3(v)/F^3(\tilde{v}) - \int_{r_2}^{\tilde{v}} [F^3(v)/F^3(\tilde{v})] dv) - \int_0^{r_2} r_2 dF^3(v)/F^3(\tilde{v}) \right] \\
= \; & \int_{r_2}^{\tilde{v}} F^3(v) dv.
\end{aligned}$$

Therefore, $\tilde{v}$ satisfies

$$(\tilde{v} - p_1)\frac{1 + F(\tilde{v})}{2} = \int_{r_2}^{\tilde{v}} F^3(v) dv. \tag{3.13}$$

When $F(v) = v$, $(\tilde{v} - p_1)(\tilde{v} + 1)/2 = \int_{r_2}^{\tilde{v}} v^3 dv$. The expected revenue is

$$\begin{aligned}
R(P_{p_1}, A_{r_2} | p_1 > r_2) \;=\; & [1 - F^2(\tilde{v})]p_1 - F^2(\tilde{v})c - F^4(r_2)r_2 \\
& + \int_0^{r_2} r_2[6(1 - F(v)) + 6(F(\tilde{v}) - F(v))] f(v) F^2(v) dv \\
& + \int_{r_2}^{\tilde{v}} v[6(1 - F(v)) + 6(F(\tilde{v}) - F(v))] f(v) F^2(v) dv \\
& + \int_{\tilde{v}}^1 v 2(1 - F(v)) f(v) F^2(\tilde{v}) dv.
\end{aligned} \tag{3.14}$$

When the seller's mechanism sequence is (P_{p_1}, A_{r_2}) with $p_1 \leq r_2$, the forward-looking buyers will not wait until next period and pay a higher price. The expected revenue

$$R(P_{p_1}, A_{r_2} | p_1 \leq r_2) = p_1[1 - F^2(p_1)] + F^2(p_1) \left[\int_{r_1}^1 \alpha(v) dF^2(v) - c \right].$$

3.6.3 Auction-Auction Sequence

Suppose the seller's mechanism sequence is (A_{r_1}, A_{r_2}) with $r_1 > r_2$. The utility of buying in the first period is

$$F(\tilde{v})\tilde{v} - F(r_1)r_1 - \int_{r_1}^{\tilde{v}} vdF(v) = \int_{r_1}^{\tilde{v}} F(v)dv.$$

The utility of not buying in the first period is

$$\int_{r_2}^{\tilde{v}} F^3(v)dv.$$

$\tilde{v}$ satisfies

$$\int_{r_1}^{\tilde{v}} vdv = \int_{r_2}^{\tilde{v}} F^3(v)dv. \tag{3.15}$$

The expected revenue is

$$\begin{aligned} R(A_{r_1}, A_{r_2}|r_1 > r_2) \;\; = \;\; & 2[1 - F(\tilde{v})]F(\tilde{v})r_1 + \int_{\tilde{v}}^{1} v2(1 - F(v))f(v)dv - c \\ & -F^2(\tilde{v})c - F^4(r_2)r_2 \\ & + \int_{0}^{r_2} r_2[6(1 - F(v)) + 6(F(\tilde{v}) - F(v))]f(v)F^2(v)dv \\ & + \int_{r_2}^{\tilde{v}} v[6(1 - F(v)) + 6(F(\tilde{v}) - F(v))]f(v)F^2(v)dv \\ & + \int_{\tilde{v}}^{1} v2(1 - F(v))f(v)F^2(\tilde{v})dv. \end{aligned} \tag{3.16}$$

If $r_1 \leq r_2$, then the first-period buyers will participate in the auction in the first period for its lower reserve price and reduced competition. The seller's revenue is

$$R(A_{r_1}, A_{r_2}|r_1 \leq r_2) = \int_{r_1}^{1} \alpha(v)dF^2(v) - c + F^n(r_1)\left[\int_{r_2}^{1} \alpha(v)dF^2(v) - c\right].$$

3.6.4 Auction-Price Sequence

Suppose the seller's mechanism sequence is (A_{r_1}, P_{p_2}) with $r_1 > p_2$. The utility from buying is

$$\int_{r_1}^{\tilde{v}} F(v)dv.$$

The utility not buying is

$$(\tilde{v} - p_2)[F^3(p_2) + F^2(p_2)(1 - F(p_2)) + \frac{1}{2}(F(\tilde{v}) - F(p_2))F^2(p_2) + \frac{1}{3}F(p_2)(1 - F(p_2))^2$$
$$+\frac{1}{3}(F(\tilde{v}) - F(p_2))2F(p_2)(1 - F(p_2)) + \frac{1}{4}(F(\tilde{v}) - F(p_2))(1 - F^2(p_2))].$$

When $F(v) = v$, $\tilde{v}$ satisfies

$$\begin{aligned}\int_{r_1}^{\tilde{v}} vdv \quad &= \quad \delta(\tilde{v} - p_2)[p_2^3 + \frac{1}{2}2p_2^2(1 - p_2) + \frac{1}{2}(\tilde{v} - p_2)p_2^2 \\ &+\frac{1}{3}p_2(1 - p_2)^2 + \frac{1}{3}2(\tilde{v} - p_2)p_2(1 - p_2) + \frac{1}{4}(\tilde{v} - p_2)(1 - p_2^2)] \qquad (3.17)\end{aligned}$$

The expected revenue is

$$R(A_{r_1}, P_{p_2}) = 2[1 - F(\tilde{v})]F(\tilde{v})r_1 + \int_{\tilde{v}}^{1} \alpha(v)dF^2(v) - c + [F^2(\tilde{v}) - F^4(p_2)]p_2. \qquad (3.18)$$

When $r_1 < p_2$, all buyers buy in the first period, and the expected revenue is

$$R(A_{r_1}, P_{p_2}) = \int_{r_1}^{1} \alpha(v)dF^2(v) - c + F^2(r_1)[1 - F^2(p_2)]p_2.$$

3.7 Infinite Horizon

Unlike goods with expirations such as airline tickets and hotel rooms or seasonal clothes that might fall out of favor in three months, many goods such as cellphones, stamps and books do not lose value and do not have fixed deadlines to be sold, but nonetheless, the seller has incentive to sell the good and realize the payment as soon as possible. In this section, the optimal sequence of mechanisms when in the infinite horizon is characterized. The solution turns out to be relatively simple and straightforward: the optimal mechanism sequence is an infinite repetition of the same mechanism. Different from the static setting, the optimal reserve price depends on not only the buyer value distribution but also the number of buyers per period and crucially the operational cost.

Although it is an infinite horizon problem, the seller faces the same problem in each period. Therefore, she only needs to solve a static problem in a stationary setting, and her optimal mechanism sequence is either an infinitely repeated auction or an infinitely repeated posted price - running a reserve price auction until some buyers bid, or posting a constant price until some buyers buy at the price. We will first calculate respectively the optimal constant price and the optimal constant reserve price and then compare to see which infinite mechanism sequence generates more expected profits. Finally, we verify that the better of the two is indeed the expected revenue maximizing mechanism in each individual period.

First, we solve the optimal posted price p_∞^*. Let π_∞^P denote the expected profit of posting p_∞^* in each period. Together, p_∞^* and π_∞^P must simultaneously satisfy the following equations,

$$\rho(p_\infty^*) = \delta\pi_\infty^P, \tag{3.19}$$

$$\pi_\infty^P = \pi(P_{p_\infty^*}) + \delta k\left(p_\infty^*\right)\pi_\infty^P, \tag{3.20}$$

where equation (3.19) equates the posted price marginal revenue to the expected profit, and equation (3.20) shows that the optimal expected profit equals the profit of using $P_{p_\infty^*}$ plus the expected profit if it is not sold. Rearranging equation (3.20) and plugging it into equation (3.19),

$$\frac{1-\delta}{1-\delta k(p_\infty^*)}p_\infty^* - \frac{1-F^n(p_\infty^*)}{[F^n(p_\infty^*)]'} = 0. \tag{3.21}$$

Since the first term is strictly increasing in and the second term is strictly decreasing in p_∞^*, and when $p_\infty^* = 0$, the LHS is negative $(= -(1 - F^n(0))/(F^n(0))')$, but when $p_\infty^* = 1$, the

LHS is positive ($=1$), p_∞^* is uniquely determined by.

Similarly, we can calculate the optimal reserve price $r_\infty^*(c)$ for a cost c seller. Let $\pi_\infty^A(c)$ denote the continuation value of running $A_{r_\infty^*(c)}$ each period. Together, $r_\infty^*(c)$ and π_∞^A must simultaneously satisfy the following two equations,

$$\begin{aligned} \alpha(r_\infty^*(c)) &= \delta\pi_\infty^A(c), & (3.22)\\ \pi_\infty^A(c) &= \pi(A_{r_\infty^*(c)}) + \delta k(r_\infty^*(c))\pi_\infty^A(c). & (3.23) \end{aligned}$$

Substitute equation (3.23) into equation (3.22) and rearrange,

$$\frac{1-\delta}{1-\delta k(r_\infty^*(c))} r_\infty^*(c) = \frac{1-F(r_\infty^*(c))}{f(r_\infty^*(c))} + \frac{\delta\left[R(A_{r_\infty^*(c)}) - R(P_{r_\infty^*(c)}) - c\right]}{1-\delta k(r_\infty^*(c))}. \tag{3.24}$$

The LHS is increasing and the RHS is decreasing in $r_\infty^*(c)$; the LHS is 0 and the RHS is $1/f(0) + \delta[R(A_0) - c]/(1-\delta)$ when $r_\infty^*(c) = 0$ and the LHS is 1 and the RHS is $-\delta c$ when $r_\infty^*(c) = 1$. Therefore, if $c \le R(A_0)$, $r_\infty^*(c)$ is uniquely determined by the system of equations. If $c > R(A_0)$, the seller chooses a posted price for sure, because for any r, A_r generates strictly less expected profit than P_r, as

$$\pi(A_r) - \pi(P_r) \le \pi(A_0) - \pi(P_0) = R(A_r) - c < 0.$$

Therefore, there is a cutoff cost c_∞^* such that the seller runs $A_{r_\infty^*(c)}$ if $c > c_\infty^*$, runs $P_{p_\infty^*}$ if $c < c_\infty^*$ and is indifferent between the two if $c = c_\infty^*$. c_∞^* must satisfy $\pi_\infty^A(c_\infty^*) = \pi_\infty^P$. There is an easy way to calculate c_∞^* than solving equations (3.23) and (3.24). $r_\infty^*(c_\infty^*)$ is determined by equation (3.23),

$$\alpha(r_\infty^*(c_\infty^*)) = \delta\pi_\infty^P, \tag{3.25}$$

but c_∞^* must satisfy that in any period, the seller is indifferent between posting the optimal price and running the optimal auction instead, so $\pi_\infty^P = R(A_{r_\infty^*(c_\infty^*)}) - c_\infty^* + \delta k(r_\infty^*(c_\infty^*))\pi_\infty^P$,

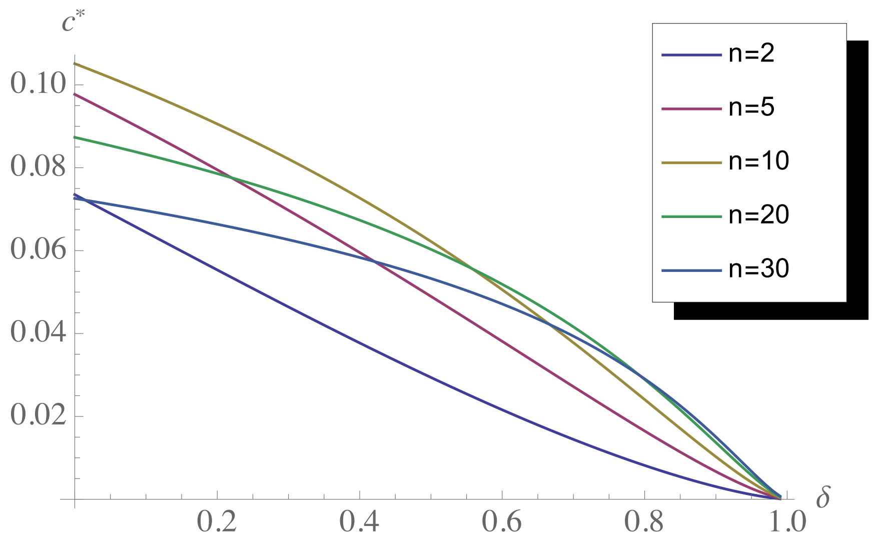

Figure 3.8: The cutoff cost for different numbers of buyers and discount factors when the buyers' values are uniformly distributed between 0 and 1.

or

$$c_\infty^* = R(A_{r_\infty^*(c_\infty^*)}) - [1 - \delta k(r_\infty^*(c_\infty^*))]\, \pi_\infty^P. \tag{3.26}$$

The optimal mechanism sequence is summarized as follows.

Proposition 14. *Suppose $T = \infty$. Let p_∞^* and π_∞^P be the unique solution to equations (3.19) and (3.20). Let c_∞^* be the unique solution to equations (3.25) and (3.26). Fix any $c \leq R(A_0)$, let $r_\infty^*(c)$ be the unique solution to equation (3.24). The seller's optimal mechanism sequence is an infinitely repeated sequence of $m_\infty^*(c)$. If $c > c_\infty^*$, then $m_\infty^*(c) = P_{p_\infty^*}$, if $c < c_\infty^*$, then $m_\infty^*(c) = A_{r_\infty(c)}$, and if $c = c_\infty^*$, $m_\infty^*(c) = P_{p_\infty^*}$ or $m_\infty^*(c) = A_{r_\infty^*(c_\infty^*)}$. Her expected payoff by running the optimal mechanism is $\pi_\infty^*(c) = \max\left\{\pi_\infty^P, \pi_\infty^A(c)\right\}$ where $\pi_\infty^A(c)$ is obtained from equation (3.22).*

Figure 3.8 shows c_∞^* for different combinations of n and δ when $v \sim \text{Unif}[0,1]$. For example, if $\delta = 0.95$, $n = 5$, $c_\infty^* \approx 0.0026$. Fixing any δ, the relationship between c_∞^* and n is non-monotonic. Notice that fixing any n, as δ increases, i.e. when the seller becomes more patient, c_∞^* decreases. When $\delta \to 1$, there is less discounting and because the seller lives forever, she essentially encounters infinitely many buyers over her life so the order statistics effect dominates. Consequently, $c_\infty^* \to 0$. The monotonic relationship between c_∞^* and δ holds in general for any F and n. An implication of the result is that if the buyers arrive over time, as δ is small, we can think of the dynamic buyer arrival as continuous and stochastic (Poisson arrival, for example). We see that as δ becomes small, c_∞^* approaches 0, so a constant price is optimal. The proof relies on the facts that as δ increases, both π_∞^P and p_∞^* strictly increase, and the optimal reserve price $r_\infty^*(c_\infty^*)$ is always smaller than the posted price p_∞^*.

Proposition 15. *Suppose $T = \infty$. When δ increases, c_∞^* strictly decreases. That is, when the seller becomes more patient, an auction becomes less attractive.*

Proof. For simplicity let $r_\infty^* \equiv r_\infty^*(c_\infty^*)$ throughout the proof. Differentiate c_∞^* with respect to δ in equation (3.26),

$$\begin{aligned}\frac{dc_\infty^*}{d\delta} &= -\alpha(r_\infty^*)k'(r_\infty^*) - [1-\delta k(r_\infty^*)]\frac{d\pi_\infty^P}{d\delta} - [-k(r_\infty^*) - \delta k'(r_\infty^*)]\pi_\infty^P \\ &= k(r_\infty^*)\left[-\alpha(r_\infty^*) + \delta\pi_\infty^P\right] + k(r_\infty^*)\pi_\infty^P - [1-\delta k(r_\infty^*)]\frac{d\pi_\infty^P}{d\delta}.\end{aligned}$$

The first term $k(r_\infty^*)\left[-\alpha(r_\infty^*) + \delta\pi_\infty^P\right] = 0$ by equation (3.25), so

$$\frac{dc_\infty^*}{d\delta} = k(r_\infty^*)\left[\pi_\infty^P + \delta\frac{d\pi_\infty^P}{d\delta}\right] - \frac{d\pi_\infty^P}{d\delta}. \tag{3.27}$$

By equations (3.19) and (3.25), and the facts that $\alpha(\cdot) \geq \rho(\cdot)$, we have $r_\infty^* \leq p_\infty^*$ and consequently $k(r_\infty^*) \leq k(p_\infty^*)$. Therefore

$$\frac{dc_\infty^*}{d\delta} \leq k(p_\infty^*)\left[\pi_\infty^P + \delta\frac{d\pi_\infty^P}{d\delta}\right] - \frac{d\pi_\infty^P}{d\delta}.$$

It suffices to show that the RHS is negative.

Rearrange equation (3.20),

$$[1-\delta k(p_\infty^*)]\pi_\infty^P = p_\infty^*[1-k(p_\infty^*)],$$

and differentiate it with respect to δ,

$$\frac{d\pi_{\infty}^{P}}{d\delta}\left[1-\delta k(p_{\infty}^{*})\right]-\pi_{\infty}^{P}\left[k(p_{\infty}^{*})+k'(p_{\infty}^{*})\right]=-p_{\infty}^{*}k'(p_{\infty}^{*})+\left[1-k(p_{\infty}^{*})\right]\frac{dp_{\infty}^{*}}{d\delta}.$$

Rearrange, we obtain

$$k\left(p_{\infty}^{*}\right)\left[\pi_{\infty}^{P}+\delta\frac{d\pi_{\infty}^{P}}{d\delta}\right]-\frac{d\pi_{\infty}^{P}}{d\delta}=\left[\rho\left(p_{\infty}^{*}\right)-\pi_{\infty}^{P}\right]k'\left(p_{\infty}^{*}\right)\frac{dp_{\infty}^{*}}{d\delta}.$$

The RHS by equation (3.19) equals $(\delta-1)\,\pi_{\infty}^{P}k'\left(p_{\infty}^{*}\right)\frac{dp_{\infty}^{*}}{d\delta}$. Differentiate equation (3.19) with respect to δ,

$$\pi_{\infty}^{P}+\delta\frac{d\pi_{\infty}^{P}}{d\delta}=\frac{\partial\rho(p_{\infty}^{*})}{\partial p_{\infty}^{*}}\frac{dp_{\infty}^{*}}{d\delta}>0.$$

$\rho'>0$ implies $dp_{\infty}^{*}/d\delta>0$, which implies $(\delta-1)\,\pi_{\infty}^{P}k'\left(p_{\infty}^{*}\right)\frac{dp_{\infty}^{*}}{d\delta}<0$ as $\delta<1$. □

3.8 Conclusion

This paper studies a monopolist seller's sequence of profit-maximizing choices between posted prices and costly auctions when buyers arrive to the market over time. Most interestingly, we find that in a variety of settings, posting a price before auctioning is desirable, but auctioning before posting a price is not. An optimal auction has a low reserve price and thus a high sale probability, so the use of an auction in an earlier period is associated with a higher probability of giving up the good. An auction incurs not only a constant fixed operational cost, it also incurs an endogenous opportunity cost of the retention value of the good. Prices-auctions sequences are more desirable than auctions-prices sequences because the retention value, the opportunity cost of selling through an auction, is decreasing over time. On eBay in particular, a buy-it-now option that allows a buyer to snatch a good for a fixed price before the seller starts an auction resembles the optimal price-auction sequence. Finally, a constant price can be optimal when the seller does not have a deadline to sell the good.

Although we have explored the following extensions to different degrees, they are still interesting and worthwhile to be explored to fuller extents. The cost of mechanisms is not a consideration in any of the papers that has been ignored in other papers, so it is worthwhile to explore how the mechanism cost and the mechanism choice can interact.

First, the seller may have multiple copies of the identical goods. We have explored the possibility that the seller sells the good sequentially, but it is worthwhile to consider other forms of auctions in which the seller can seller more than one good at a time. The dynamic programming problem carries one more state variable besides the seller's age - the number of remaining objects she has. Given the complexity of the problem with even one good, the multiple-goods extension should be treated as a separate pursuit complementary to the current one.

Another nontrivial and interesting extension is to allow the buyers to be long-lived and strategic in choosing their purchase times and may exit randomly. We have shown that the main results hold in a numerical example. It is important to explore how the result holds in general, when there are more than two periods, more than two buyers, and with general value distributions. I suspect that the optimality of prices-auctions sequence and the sub-optimality of the auction-prices sequence remain. The main difficulty is that we cannot solve the problem period-by-period, and the optimal profit does not necessarily decrease over time because new buyers arrive and old buyers stay. Deb and Pai (2012) and Pai and Vohra (2013) study this problem in a costless environment and show that simple index rules and ironing can be optimal. Said (2012) and Li (2009) show that open ascending auctions are suitable for perishable and storable goods, respectively, when the buyers arrive over time and stay and strategically time their purchases.

Finally, the seller has been assumed to commit to a particular sequence of mechanisms. The seller's commitment to future mechanisms. Dilme and Li (2012) consider the equilibrium non-committal price paths of the seller. Skreta (2006) shows that simple prices can be optimal when the seller does not have commitment power.

Chapter 4 Prices and Auctions in Dynamic Markets

Abstract: This paper studies the dynamic markets in which infinitely many unit-supply short-lived sellers and unit-demand long-lived buyers with persistent private values are matched according to a frictional process. Although reserve price auctions are expected revenue maximizing in the unique steady state equilibrium, the optimal revenue decreases when the market becomes more consumer friendly, namely when the buyers survive longer, face fewer bidding competitors, and become more patient. In particular, the revenue advantage from running an auction with a reserve price over simply posting that reserve price reduces as the market becomes more buyer friendly, making sellers more likely to use posted prices whose prevalence however results in market-wide externalities and inefficiencies.

Keywords: optimal mechanism, reserve price auction, posted price, auction premium, equilibrium existence

JEL: D44

4.1 Introduction

In theory and practice, a standard auction with a well-chosen reserve price has shown to be desirable revenue-enhancing choice for unit-supply sellers facing buyers with independent private valuations. Although this paper confirms theoretically the revenue optimality of the reserve price auction in dynamic markets, the optimal revenue decreases when the market becomes more competitive for the sellers. More importantly, the revenue advantage relative to a simple posted price diminishes in such competitive environment.

We study steady states of the dynamic markets where infinitely many short-lived sellers and long-lived buyers who have possibly many chances to obtain a good are matched uniformly randomly. When the sellers have no fixed cost of running any mechanism, standard reserve price auctions are expected revenue maximizing with the strictly positive reserve prices indicating market competitiveness. The market becomes more consumer friendly when the buyers survive longer, face fewer competitors and become more patient, and the equilibrium reserve price and optimal auction revenue are lower in consumer friendly markets where the sellers have less market power. We generalize the monopolist's problem of Myerson (1981) by introducing a sequential market in which buyers may have opportunities to buy a perfect substitute of the good, and generalize the auction marginal revenue of Bulow and Roberts (1989) in this setting.

Although an auction with an optimally chosen reserve price generates the most expected revenue, we witness the prevalent and increasing uses of alternative mechanisms in many markets of goods with close substitutes because they have other operational advantages while achieving revenues sufficiently close to be optimal. For example used auto dealers use sequential bargaining followed by secret reserve price auctions (Larsen, 2013), treasury bill sales implement multi-unit auctions and price menus, and alternating bargaining are prevalent in housing markets. The effects of sequential competition offer a particularly plausible explanation to the rises of alternative mechanisms, in particular, the posted prices which rapidly take over eBay market (Einav et al., 2013).

Posted price mechanisms have the desirable properties of immediacy and simplicity compared with auctions which involve active organization of sellers and participation of buyers over a longer period of time. The gain in revenue from auction caused by buyer competition is depressed when the market becomes consumer friendly; more precisely, the difference between the expected revenue of the reserve price auction and that of simply posting the reserve price as the price, decreases as distribution of the buyers' potential price offers improves first-order stochastically in favor of the buyers. This result resonates that the choice between an auction and posted price depends on steepness of the marginal revenue curve in Wang (1993). When a buyer has multiple sequential opportunities to purchase a good, her willingness-to-pay to each seller is driven below her intrinsic value for the good, consequently affecting expected revenues of different sales mechanisms and possibly altering a seller's profit-maximizing choice if there is higher fixed cost associated with running an auction.

We also consider the dynamic markets with simultaneous existence of posted prices and auctions. When sellers have heterogeneous costs of running auctions, the sellers may switch to simpler mechanisms involving posting prices. The sellers who live for one period post prices, and those who can live longer use Buy-It-Now options and dynamically adjust prices, while auction followed by a posted price if it is not sold is never profit-maximizing. There are inefficiencies associated with posted prices, however, as it is less likely to be sold by seller, and it is not guaranteed that the buyer with the highest value receives the good. These inefficiencies result in market-wide externalities to future sellers and buyers.

The frictional matching process distinguishes the current work from the simultaneous auctions literature which studies competing sellers who simultaneously announce mechanisms to attract buyers to participate. In equilibrium, with mild technical conditions, each seller runs auction with reserve price lower than that in the monopoly case (Burguet and Sákovics, 1999; Pai, 2009), and as the number of sellers approaches infinity, the equilibrium mechanism is the efficient auction (McAfee, 1993; Peters and Severinov, 1997). Although the literature yields beautiful technical results, it is hardly applicable to real world: not all the agents can meet one another because of geographic or time constraints, and lower reserve price is a result of buyers' sequential opportunities rather than a tool to attract buyer participation.

The sale inefficiency that buyers who value the goods more than the sellers do not receive the goods is a result of the frictional matching process. Similar frictional matching models explain nontrivial prices in bargaining (Rubinstein and Wolinsky, 1985) and auctions (Wolinsky, 1988), as well as inevitable natural unemployment (Diamond and Maskin, 1979) despite of excess demand or supply. Another related paper, Satterthwaite and Shneyerov (2008), shows that the transaction price converges to the competitive Walrasian price.

The idea that sequential opportunities reduce buyers' willingnesses-to-pay is explored in Said (2011). It considers a dynamic setting where bidders randomly arrive and incorporate beliefs of current and future dynamics in their bids shading in efficient second price auctions. However, it takes a partial equilibrium approach by assuming that the sellers are restricted to run efficient auctions in addition to that the buyers have value independent outside options as they receive new value draws for each imperfect substitute, an assumption maintained in Wolinsky (1988) as well. This paper studies a general equilibrium when buyers have value dependent outside options so that in equilibrium the buyers with higher valuations have better opportunities and exit the market faster.

In summary, the main contributions of the paper are three-fold. First, it characterizes the revenue-maximizing mechanism when buyers have sequential outside options. Second, equilibrium of the dynamic settings where buyers have value dependent outside options, previously deemed infeasible to solve, is uniquely characterized with definitive comparative statics results which offer guidance to auction platform design. Finally, analogues are drawn between posted price mechanisms and reserve price auctions with their revenues directly compared, and it is probably the first paper to consider simultaneous existence of posted prices, auctions, and Buy-It-Now options to investigate their welfare effects in dynamic settings.

The paper is organized as follows. Section 4.2 introduces the basic dynamic setup and defines the steady state solution concept. Section 4.3 solves a generalized monopolist's problem when buyers have sequential outside options and examines the effects of changes in the outside options. Section 4.4 characterizes the stationary equilibrium and Section 4.5 presents comparative static results and their market implications. Section 4.6 introduces the posted price mechanism with its revenue compared to auction's and Section 4.7 discusses the welfare effects when posted prices are used. Section 4.8 concludes.

4.2 The Setup

In this section, we describe the dynamic setup and define the equilibrium. There are countably infinite periods labeled by $t = 0, 1, \cdots$. In the beginning of each period, there are measure 1 continuum of unit-supply sellers and integer measure n continuum of unit-demand buyers in a homogeneous good market. We refer to n as the **buyer-seller ratio**. All the agents are risk-neutral, expected utility maximizers with quasilinear preferences, and have common discount factor δ for the next period. Each seller lives for one period and has normalized value zero for the good. Each buyer possibly lives forever and has persistent private value v independently and identically drawn from value distribution F with positive density f on support $[0, 1]$. Let H_t denote period t's active buyer value distribution composed of old and newborn buyers, except for that in period 0 when all the buyers are newborn ($H_0 = F$).

The market proceeds as follows. Buyers and sellers are randomly matched. Sellers each choose a sales mechanism, and buyers participate in the mechanisms. The number of buyers each seller is matched with is n.

Seller s chooses a direct anonymous mechanism (DAM) $M_{s,t}$ in which the buyers cannot be distinguished by their ages or identities but only by reported values. It consists of probability assignment and cost functions $\{P_i(\cdot), C_i(\cdot)\}_i$ which, for any vector of value reports $\mathbf{z} = (z_1, \cdots, z_n)$ specifies each buyer's probability of winning and payment. Therefore, A buyer i's expected probability of winning and expected payment by reporting z_i are

$$\begin{aligned} \overline{P}_i(z_i) &= \int_{\mathbf{z}_{-i}} P_i(z_i, \mathbf{z}_{-i})\, dH_{-i}(\mathbf{z}_{-i}), \\ \overline{C}_i(z_i) &= \int_{\mathbf{z}_{-i}} C_i(z_i, \mathbf{z}_{-i})\, dH_{-i}(\mathbf{z}_{-i}) \end{aligned}$$

where $H_{-i}(\mathbf{z}_{-i}) = \prod_{j=1}^{n} H_j(z_j)$.

A buyer i of value v, after knowing what mechanism $M_{s,t}$ she participates in, plays a feasible strategy $\sigma_{i,t}(v, M_{s,t}) \in [0,1]$. If buyer i obtains the good from the seller she is matched with, she is a **winner** ($i \in W_t$) and exits the market; otherwise, she is a **loser** ($i \in L_t$) and survives with probability s to period $t+1$, and we refer to s as the **survival rate**. Newborn buyers enter the market in the beginning of period $t+1$ to replace the winners and exiting losers to to keep a constant measure n of buyers.

Let $l_{i,t}(v|\sigma_{i,t}(v, M_{s,t}), \sigma_{-i,t}(\cdot, M_{s,t}), M_{s,t})$ denote value v buyer i's expected probability of losing when she plays $\sigma_{i,t}(v, M_{s,t})$ and her opponents play $\sigma_{-i,t}(\cdot, M_{s,t})$ in the mechanism $M_{s,t}$ she participates in, and $l_t\left(\{\sigma_i(\cdot,\cdot)\}_i, \{M_{s,t}\}_s\right)$ is the proportion of losers in period t given the agents' behaviors. We subsequently denote them $l_{i,t}(v)$ and l_t, but keep in mind that they depend on behavior of the active agents. Period $t+1$ active buyers are composed of newborn buyers and surviving losers from period t, so the expected active value distribution in period $t+1$ is

$$\mathbb{E}\left[H_{t+1}(v)|H_t\right] = (1 - sl_t) F(v) + sH_t(v|L_t) \tag{4.1}$$

where $H_t(v|L_t)$ is the value conditional distribution for losing buyers.

Before defining the equilibrium, we shall comment on some aspects of the model. Wolinsky (1988) employed similar frictional matching technology reflecting information and search frictions. The stochastic matching however adds no further insights but computational complexities, so we restrict our attention to the uniform matching with deterministic buyer-seller ratio. Furthermore, a seller can hold multiple goods in the same period, and as long as he does not own a positive measure of the goods in the market, his actions cannot alter the subsequent market compositions.

Two tunnels of learning arise in finite markets but not in infinite and anonymous ones. In a market with finite number of buyers, there is significant probability that a buyer's opponents may become her opponents again in subsequent periods, so when the transaction price is not announced in an auction, a bidder can infer from her bid about the possible transaction price and values of her opponents. In a market with finite number of items for sale (sellers), a buyer knowing whether the item from the mechanism he has participated in is sold or not helps her to learn about nontrivial portion of the market. Information asymmetry arises in sellers facing markets of asymmetric bidders knowing different amounts of information from history of actions and outcomes despite of symmetric newborn distributions. Milgrom and Weber (2000) studies the setting of n buyers, and one seller with $k \leq n$ goods and shows these learning effects make the expected price path a martingale. Said (2012) shows that a simultaneous ascending auction can alleviate these effects resulted from stochastic arrival of the agents.

Continuum of agents and restrictions to the direct anonymous mechanisms guarantee information symmetry among all the participating agents. Numerical results show that the learning effect is relatively small and vanishes rather quickly even in medium-sized markets[1]. Furthermore, Bodoh-Creed (2012) shows that the equilibrium with continuum of agents is approximated by that of finite markets. In this particular setting, the sellers have symmetric beliefs about buyers' values. Anonymous mechanisms further restrict sellers to discriminate by the buyers' ages that potentially reveal buyers' losing histories and thus values. Learning by bidding is also not plausible as the online market is so enormous that no buyer aims at beating any particular opponent.

The assumption that masses of agents stay constant is nonrestrictive as well. Suppose that the market size grows each period so that masses of sellers and buyers both increase by $g \geq 0$. This market growth effect is actually equivalent to decreasing the buyer's survival rate by $1/\left(1+g\right)$. Instead of sl_t, there is only $sl_t/\left(1+g\right)$ of surviving losers in period $t+1$.

[1]The beneficial probability of learning from bidding and outcome is of order $1-\left(1-s/k\right)^{n-1}$ where k is the total number of sellers. It vanishes rather quickly in market of small size with relatively low survival rate s. For example, for $s=1/2$, $k=10$ and $n=5$, the probability of a buyer encountering a previous opponent in the next period is 20%, conditional on her own survival.

The active value distribution is then

$$\mathbb{E}\left[H_{t+1}\left(v\right)|H_t\right] = \left(1 - \frac{s}{1+g} l_t\right) F\left(v\right) + \frac{s}{1+g} H_t\left(v|L_t\right).$$

When $s' = s/\left(1+g\right)$, the buyer value evolution coincides with (4.1). Although each buyer survives with probability s, the market growth rate essentially makes the survival rate $s' = s/\left(1+g\right)$.

For the remainder of the section, we define the equilibrium concept we want to solve, the stationary symmetric sequential equilibrium (SSSE) where buyers and sellers of the same value play the same utility maximizing strategies in each period when the expected active buyer value distribution remains stationary, and the belief of the buyers and sellers updates according to the equilibrium behavior.

Agents' strategies and beliefs are defined as follows. Since all the sellers are identical and the buyers symmetric, we restrict our attention to symmetric behavior where all the sellers chooses the same mechanism M_t and each value v buyer behaves according to strategy $\sigma_t\left(v, M_t\right)$; that is, agents are only distinguished by their types but not by indices. In period t, each agent has belief $\mu_t\left(\{H_{t'}, M_{t'}, \sigma_{t'}\}_{t'=t}^{\infty}\right)$ on the active buyer value distribution, the seller mechanism choice and the buyer strategies for the current and every subsequent period.

The payoffs of the agents are specified as follows. Each seller gets expected revenue $r\left(M_t|\mu_t\right)$ by choosing mechanism M_t under belief μ_t. Each value v buyer receives total discounted expected utility $u\left(v, \sigma_t\left(v, M_t\right), \sigma_{-i,t}\left(\cdot, M_t\right)|M_t, \mu_t\right)$ when her $n-1$ opponents play $\sigma_{-i,t}\left(\cdot, M_t\right)$.

Subsequent active buyer value distributions $\mathbb{E}\left[H_{t+1}\left(v\right)|H_t, M_t, \sigma_t\right]$ can be inferred from the buyer and seller behavior and pre-specified market and matching dynamics. If the active buyer value distribution is stationary, then the buyer composition will be expected to be the same across periods, so the stationary value distribution is $H_*\left(v\right) \equiv \mathbb{E}\left[H_t\left(v\right)|\mu_*\right] = \mathbb{E}\left[H_{t+1}\left(v\right)|\mu_*\right]$.

In particular, given the equilibrium mechanism M_* and strategy $\sigma_*(\cdot,\cdot)$, the stationary value PDF is the weighted average of newborn PDF f and the previous period PDF h_* in stationary equilibrium. The stationary value PDF and CDF are respectively

$$\begin{aligned} h_*(v) &= (1-sl_*)\, f(v) + sl_*(v)\, h_*(v) && (4.2)\\ H_*(v) &= (1-sl_*)\, F(v) + s\int_0^v l_*(z)\, dH_*(z) && (4.3)\end{aligned}$$

where $l_*(\cdot)$ and l_* depend on the equilibrium seller and buyer behaviors and matching dynamics.

Now we can define the equilibrium where 1) all the agents play symmetric stationary strategies, 2) the active value distribution is stationary, 3) agents' beliefs about the equilibrium behavior and active value distribution are correctly updated, and 4) every agent maximizes the expected payoff given beliefs.

Definition 6. $(M_*, \sigma_*, H_*, \mu_*)$ constitutes a **stationary symmetric sequential equilibrium (SSSE)** where every seller runs the same mechanism M_* and every value v buyer plays according to strategy $\sigma_*(v, M)$ in mechanism $M \in \mathcal{M}$ and it satisfies the following conditions.

1. **(Stationarity)** Stationary value distribution is a martingale with respect to the equilibrium mechanism M_* and strategy σ_*:

   $$H_*(v) = \mathbb{E}\left[H_{t+1}(v)\,|H_t = H_*, M_*, \sigma_*\right],$$

   where the stationary value PDF and CDF are determined by (4.2) and (4.3), respectively.

2. **(Consistency)** Every agent has the correct belief μ_* about the stationary value dis-

tribution, equilibrium mechanism and strategy:

$$\mu_* (H_*, M_*, \sigma_*) = 1.$$

3. **(Sequential Rationality of Seller)** M_* maximizes seller's expected profit with respect to belief μ_*: $\forall M \in \mathcal{M}$, $\pi (M_* | \mu_*) \geq \pi (M | \mu_*)$.

4. **(Sequential Rationality of Buyer)** $\sigma_* (\cdot, M)$ is a symmetric Bayes-Nash equilibrium in every mechanism M: $\forall v$, $\forall M \in \mathcal{M}$, $\forall \sigma (\cdot, M) \in \Sigma_M$,

$$u (v | M, \mu_*) \geq u \left(v, \tilde{\sigma} (v, M), \sigma_{-i,*} (\cdot, M) | M, \mu_* \right),$$

where $u (v | M, \mu_*) \equiv u \left(v, \sigma_* (v, M), \sigma_{-i,*} (\cdot, M) | M, \mu_* \right)$ denotes value v buyer's equilibrium expected payoff in M.

Note in particular that the symmetry assumption does not need to be assumed but arises as an equilibrium outcome. Since all the optimal incentive-compatible mechanisms yield the same revenues and the same expected buyer payoffs, all the sellers will choose the same mechanism in equilibrium and all the buyers will have the same expected equilibrium payoffs.

Characterizing the equilibrium requires 1) characterizing the stationary value distribution, 2) solving the monopolist profit-maximizing problem with value-dependent outside options, and 3) solving the buyers' equilibrium strategies when they have sequential opportunities. In essence, in the dynamic setting, the existence of sequential markets affects both seller mechanism choice and buyer behavior, and the sequential market conditions depend on buyer composition H_*.

In Section 4.3, we solve a general version of the monopolist's problem and specifies the buyer's equilibrium strategy in the optimal mechanism and the agents' expected equilibrium payoffs. The SSSE is characterized and proven to uniquely exist in Section 4.4 and comparative statics results on the survival rate, the buyer-seller ratio and the discount factor are presented in Section 4.5.

4.3 Seller's Problem Generalized

In this section, we study revenue maximization of a monopolist who faces symmetric buyers in the presence of a sequential market where the identical good is expected to be sold by other sellers. We show that the optimal mechanism is implementable by a standard reserve price auction, but the reserve price changes with respect to the sequential market condition. In particular, the reserve price is always lower than the monopoly reserve price in Myerson (1981), an extreme setting of the current setup which incorporates the monopolist problem in the SSSE as a special case. We generalize the marginal revenue curve defined by Bulow and Roberts (1989) to this sequentially competitive setting to determine optimal reserve prices. The general revenue maximizing mechanism with ex-ante *a*symmetric buyers and related results regarding incentive-compatibility, buyer indifference and revenue-equivalence of different mechanisms are derived and presented in Appendix 4.9.

Suppose that there are n buyers and one seller which we call a monopolist. In addition to the assumptions defined above, the value distribution is assumed to have decreasing inverse hazard rate.

Assumption 2. *$d\eta\left(F\left(v\right)\right)/dv \leq 0$ where $\eta\left(F\left(v\right)\right) \equiv \left(1-F\left(v\right)\right)/f\left(v\right)$.*

In particular, the setting so far is the same as Myerson (1981), which we refer to as the **pure monopoly** setting. We introduce a sequential market where each buyer is expected to receive a price offer drawn from distribution Λ with support $[\underline{x}, \overline{x}]$ and density λ. Since Λ incorporates all the information about the subsequent period, we simply refer to the **sequential market** by the distribution Λ. Not to be interrupted by technical details, we assume that the buyers are symmetric and the sequential market distribution is continuous and differentiable.

The presence of a sequential market incorporates a wide range of settings. Myerson (1981) solves the monopolist's problem when there is no price offer (lower than 1) in the sequential market ($\Lambda(x) = 0 \forall x \leq 1$), or the buyers discount the sequential market infinitely ($\delta = 0$). In the SSSE, a buyer's continuation payoff is her total discounted expected utility from all mechanisms she participates in the future periods. Any outcome she faces can be characterized by the expected payment with the probability of such payment. The set of outcomes can be summarized by a probability distribution, thus a sequential market. In particular, the sequential opportunities each buyer faces in the stationary equilibrium are captured by an equilibrium sequential market. In addition, the sequential market can be viewed as signals buyers receive to update their valuations of the good. The dynamic mechanism design literature takes this particular stance and yield analogous results as this current section (Esö and Szentes (2007)).

Although all the buyers face the same sequential market, they may receive different realized price offers and may make different decisions based on their own valuations even if the realized price offers are the same. A buyer purchases the good if and only if her value exceeds the realized price offer she receives, so a value v buyer's expected payoff in the sequential market Λ is

$$\underline{u}_{\Lambda}(v) = \int_{\underline{x}}^{v} (v - x)\, d\Lambda(x) = \int_{\underline{x}}^{v} \Lambda(x)\, dx.$$

It increases whenever the buyers receive lower price offers with higher probability, so we say that Λ is more buyer-friendly than $\tilde{\Lambda}$ if Λ is first-order stochastically dominated by $\tilde{\Lambda}$.

Definition 7. A sequential market is **more buyer-friendly** than another if and only if it is first order stochastically dominated by the other: $\Lambda \succeq_{\mathrm{B}} \tilde{\Lambda}$ if and only if $\Lambda(x) \geq \tilde{\Lambda}(x) \forall x$.

A value v buyer will not pay more than the expected utility if she waits, so her **willingness-**

to-pay (WTP) to the monopolist is

$$w_\Lambda(v) = v - \delta \underline{u}_\Lambda(v),$$

and it is differentiable, weakly increasing and weakly concave[2]. The monopolist's problem with a sequential market corresponds to a monopolist's problem with buyers' values transformed according to an increasing, concave function. Conversely, for any concave value transformation $\tilde{v}(\cdot)$, there is a sequential market Λ that induces a WTP function $w_\Lambda(\cdot) = \tilde{v}(\cdot)$.

In essence, the monopolist maximizes expected revenue with respect to n buyers who have WTP $w(v)$ drawn from the **WTP distribution** $\tilde{F}_\Lambda(\tilde{v}) = F\left(w_\Lambda^{-1}(\tilde{v})\right)$ and $\tilde{f}_\Lambda(\tilde{v}) = f\left(w_\Lambda^{-1}(\tilde{v})\right)/w'\left(w_\Lambda^{-1}(\tilde{v})\right)$, so by Myerson (1981), the optimal mechanism is to run a standard auction with reserve price determined by a transformed virtual utility curve. The virtual utility in this setting is defined as the **competitive auction marginal revenue (MR) in the presence of sequential market** Λ

$$\mathrm{MR}_\Lambda^A(v) = w_\Lambda(v) - \eta\left(\tilde{F}_\Lambda(w_\Lambda(v))\right) = w_\Lambda(v) - \eta(F(v))\, w'_\Lambda(v). \tag{4.4}$$

In section 7, we will define the competitive posted price marginal revenue and compare those with the auction's when the sequential market changes to derive results regarding changes in revenues of different mechanisms and provide further economic insights, especially parallels between auctions and posted prices. Since WTP is increasing and concave, and inverse hazard rate is decreasing, the competitive auction marginal revenue is continuously increasing, so the optimal reserve type exists and is unique.

Proposition 16 (The Optimal Mechanism)**.** *Let $A(\rho)$ represent a standard auction that implements reserve price $w_\Lambda(\rho)$ in the presence of sequential market Λ. The revenue-maximizing mechanism in the presence of the sequential market Λ is $A_\Lambda^* \equiv A(\rho_\Lambda^*)$ where $MR_\Lambda^A(\rho_\Lambda^*) = 0$.*

[2] $w'_\Lambda(v) = 1 - \delta \underline{u}'_\Lambda(v) = 1 - \delta\Lambda(v) \geq 0$ and $w''_\Lambda(v) = -\delta \underline{u}''_\Lambda(v) = -\delta\lambda(v) \leq 0$.

Not all the buyers who have values above the reserve price will participate in the auction, but the reserve price screens for participation of buyers with willingness-to-pay above it, who have values above ρ which we call the **the reserve type**. Without the sequential market, a buyer's value is her WTP, so the pure monopoly reserve price and reserve type coincide. When there is a nontrivial sequential market, the optimal competitive reserve price is lower than the pure monopoly reserve price. The importance of determining the reserve type is its relation with the probability of sale, $1 - F^n(\rho)$: the bigger the optimal reserve type, the lower the probability of sale and the higher the expected sale efficiency to transform the good from seller to buyers.

Proposition 17. *The optimal competitive reserve price in the presence of a sequential market is lower than the optimal pure monopoly reserve price.*

Remark 1. The optimal competitive reserve price is equal to the pure monopoly reserve price when there is no price offer (weakly) lower than the pure monopoly reserve price in the sequential market, because the competitive auction marginal revenue curve is unchanged for values below the pure monopoly reserve price.

The following numerical example illustrates that we can not compare optimal reserve type across different markets as a more buyer-friendly sequential market does not guarantee a lower optimal reserve price or reserve type.

Example 2. Suppose there are two buyers ($n = 2$) with values drawn uniformly from $[0, 1]$ ($F(x) = x$). In the first market, they may receive a uniform price offer from 0 to 1 ($\Lambda_1(x) = x$), and in the second market, they may receive a lower price offer with lower probability ($\Lambda_2(x) = 2x^2 \forall x \in [0, 0.5]; x \forall x \in [0.5, 1]$). Therefore, the first market is more buyer-friendly. However, the optimal reserve type and price are higher in the first market ($\rho^*_{\Lambda_1} \approx 0.4227 > \rho^*_{\Lambda_2} \approx 0.4221$ and $w_{\Lambda_1}\left(\rho^*_{\Lambda_1}\right) \approx 0.333 > w_{\Lambda_2}\left(\rho^*_{\Lambda_2}\right) \approx 0.303$).

Finally, we can calculate the buyers' equilibrium strategies and payoffs in the standard auctions in the presence of a sequential market. The first price and second price auctions (as well as Dutch and English, and all-pay auctions) with the same reserve prices generate the same revenues, from the generalized revenue equivalence and buyer indifference result (Corollary 1), a corollary of the characterization of incentive compatible mechanisms (Proposition 26), and both are presented in Appendix 4.9. In the second price auction, buyers bid their WTPs rather than their values (which are equivalent in the setting without sequential market). In the first price auction, buyers continue to shade their bids, but according to their WTPs as well.

Proposition 18 (Equilibrium in Standard Auctions)**.** *Consider a standard reserve price auction $A(\rho)$ with n buyers in the presence of Λ. A value $v \geq \rho$ buyer in the second price auction bids her willingness-to-pay $w_\Lambda(v)$. In the first price auction, the equilibrium bidding strategy of buyer $v \geq \rho$ is*

$$\sigma(v, A(\rho)) = w_\Lambda(v) - \int_\rho^v F^{n-1}(z)\, dw_\Lambda(z) / F^{n-1}(v)\,. \tag{4.5}$$

A value v buyer's total discounted payoff is

$$u(v|A(\rho)) = \int_\rho^v F^{n-1}(z)\, dw_\Lambda(z) + \delta \underline{u}_\Lambda(v)\,, \tag{4.6}$$

and the seller's expected revenue is

$$r(A(\rho)) = \int_\rho^1 MR_\Lambda^A(v)\, dF^n(v)\,. \tag{4.7}$$

These equilibrium characterizations, along with the characterization of the optimal auction, directly apply to the dynamic setting.

4.4 Stationary Symmetric Sequential Equilibrium

In this section, we characterize the stationary symmetric sequential equilibrium and prove its existence and uniqueness under mild technical conditions. Based on the results from the previous section, every seller runs the same reserve price auction. Higher value buyers are more likely to win the auctions and to exit the market, resulting in lower value buyers staying longer and a stationary value distribution that is first order stochastically dominated by the newborn value distribution. Such an equilibrium exists and is unique under mild conditions, and convergence of equilibrium from initial period is also discussed.

The equilibrium is characterized by the optimal reserve type and the stationary distribution. Since the buyers' sequential options in the SSSE can be summarized by a sequential market Λ_* which we will characterize, Proposition 16 shows the optimal mechanism is a standard auction with reserve type ρ_* determined by the equilibrium auction MR,

$$\mathrm{MR}_*^{\mathrm{A}}\left(\rho_*\right) \equiv w_*\left(\rho_*\right) - \frac{1 - H_*\left(\rho_*\right)}{h_*\left(\rho_*\right)} w_*'\left(\rho_*\right) = 0,$$

where $w_*\left(v\right)$ is value v buyer's equilibrium WTP. However, since the buyers' realized payments are all higher than ρ_*, by Remark 1, and the equilibrium reserve type is the same as the equilibrium reserve price, and is determined by

$$\rho_* - \frac{1 - H_*\left(\rho_*\right)}{h_*\left(\rho_*\right)} = 0. \tag{4.8}$$

The equilibrium reserve price is always positive, in contrast to the result obtained by McAfee (1993) that it converges to zero. As results from the next section demonstrate, the equilibrium reserve price varies with different parameters of the model, indicating level of market competitiveness. These competitive forces alleviate the frictions in matching by reducing the equilibrium reserve price to be close to the efficient level. The sellers are able to set nontrivial prices even if there is no excess demand which resonate the natural unemployment resulted from frictional search and matching models (Diamond and Maskin, 1979). When $n = 1$, this setting is a dynamic bargaining setting with equal measure of buyers and sellers, the model is similar to Rubinstein and Wolinsky (1985).

As a result a value v buyer wins the auction if and only if her value is above ρ_* and no opponent has value above v, so her probability of winning is $\overline{P}_*(v) = 1 - \mathbf{1}_{v>\rho_*} H_*^{n-1}(v)$. Conversely, a value v buyer's losing probability $l_*(v)$ is 1 if $v \le \rho_*$ and $1 - H_*^{n-1}(v)$ if $v > \rho_*$. Equilibrium loser proportion is then

$$l_* = \int_0^1 l_*(v)\, dH_*(v) = 1 - \int_{\rho_*}^1 H_*^{n-1}(v)\, dH_*(v) = 1 - \frac{1}{n}\left(1 - H_*^n(\rho_*)\right).$$

In order words, measure $1 - H_*^n(\rho_*)$ of the sellers sell their goods.

The stationary value distributions can be determined by 4.2 and 4.3 by substituting for $l(v) = 1 - \overline{P}(v)$, a value v buyer's expected probability of losing in a direct anonymous mechanism. Value v buyer's expected utility of reporting z is $\overline{P}(z)v - \overline{C}(z)$ in the mechanism and is $\overline{P}_*(v)v - \overline{C}_*(v)$ if she does not win. The stationary value distributions are determined as follows.

$$h_*(v) = \left[1 - s\int_0^1 \left(1 - \overline{P}_*(z)\right) dH_*(z)\right] f(v) + s\left(1 - \overline{P}_*(v)\right) dh_*(v) \tag{4.9}$$

$$H_*(v) = \left[1 - s\int_0^1 \left(1 - \overline{P}_*(z)\right) dH_*(z)\right] F(v) + s\int_0^v \left(1 - \overline{P}_*(z)\right) dH_*(z) \tag{4.10}$$

Given the equilibrium losing probability and proportion of losers, the stationary value distributions characterized by (4.2) and (4.3) are

$$h_*(v) = \left[1 - s\left(1 - \frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right)\right] f(v) \Big/ \left(1 - s\left(1 - \mathbf{1}_{v>\rho_*} H_*^{n-1}(v)\right)\right), \tag{4.11}$$

$$H_*(v) = \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] F(v) - \mathbf{1}_{v>\rho_*}\frac{s}{1-s}\frac{1}{n}\left(H_*^n(v) - H_*^n(\rho_*)\right). \tag{4.12}$$

In particular, the stationary distribution at the equilibrium reserve type is determined by

$$\frac{h_*(\rho_*)}{f(\rho_*)} = \frac{H_*(\rho_*)}{F(\rho_*)} = \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right]. \tag{4.13}$$

As a consequence of the auction's allocative efficiency that the buyer with the highest value above the reserve type is allocated the good, higher value buyers exit the market faster, resulting in a stationary value distribution that is first order stochastically dominated by the newborn distribution[3].

In summary, $(\mathrm{A}(\rho_*), \sigma_*, H_*, \mu_*)$ constitutes a SSSE. Each seller runs reserve price auction $\mathrm{A}(\rho_*)$ with ρ_* determined by the system of equations, (4.8) and (4.13) which also pin down $H_*(\rho_*)$. The stationary value distribution H_* is characterized by (4.8) and (4.12) given $H_*(\rho_*)$. Each buyer reports truthfully in each incentive-compatible direct anonymous mechanism (bids $w_*(v)$ in second price auction by Proposition (18)) and the equilibrium belief is the same across periods, $\mu_*(\mathrm{A}(\rho_*), \sigma_*, H_*) = 1$.

Such an equilibrium exists when losing buyers exit the market sufficiently fast. When enough higher value newborns enter the market to ensure the active value distribution with decreasing inverse hazard rate, the reserve price auction remain optimal. The exact sufficient assumptions are as follows.

Assumption 3. *The newborn value distribution is convex:* $F''(v) \geq 0$.

Assumption 4. *The survival rate is sufficiently small:* s *and* δ *satisfy* $\delta s^2 - 2s + 1 \geq 0$. *Equivalently,* $s \leq \left(1 - \sqrt{1-\delta}\right)/\delta$ *for all* δ.

These are reasonable assumptions. Although convex distribution assumption is more restrictive than that of monotone hazard rate, it includes some standard distributions such

[3]Rearrange (4.12),

$$\frac{H_*(v)}{F(v)} = \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] / \left[1 + \mathbf{1}_{v>\rho_*}\frac{s}{1-s}\frac{1}{n}\frac{H_*^n(v) - H_*^n(\rho_*)}{H_*(v)}\right].$$

Since $\left[H_*^n(v) - H_*^n(\rho_*)\right]/H_*(v)$ achieves its maximum $1 - H_*^n(\rho_*)$ at $v = 1$, RHS is bigger than 1.

as uniform and exponential distributions. Although in this paper we do not have a buyer entry stage, the equilibrium distribution resulted from a reasonable entry model should be one where higher value buyers are more likely to enter. The assumption on the survival rate and discount factor is not restrictive. First of all, as long as $s \leq 1/2$, the equilibrium exists regardless of δ. And for higher discount factor, the range of survival rate that supports a unique equilibrium becomes larger. For example, for $\delta = 0.95$, $s \leq 0.876$ satisfies the condition. Furthermore, note that these assumptions are by means necessary conditions for existence or uniqueness but rather loose sufficient conditions. Finding the necessary conditions for equilibrium existence is not key to the arguments of the paper. Furthermore, they are conditions that not only guarantee equilibrium existence but also uniqueness.

Proposition 19. *When Assumptions 3 and 4 hold, there exists a unique SSSE.*

This equilibrium is reached from periods of plays where the reserve price monotonically decreases. Starting with the initial period in which all the buyers are newborns, the sellers post auctions with a reserve price higher than the equilibrium one, because buyers value distribution FOSDs the stationary distribution and the buyers face worse sequential market than the buyers in the steady state. However, as the time progresses, lower value buyers congest the market and expect more buyer-friendly environment in the future periods, so the sellers post lower reserve price in response. The sequence of reserve price monotonically converges to the equilibrium reserve price. The monotonicity of adjusted inverse hazard rate is guaranteed because the sequential market improves and the active buyer distribution decreases as higher value buyers win more often and exit faster.

The equilibrium sequential market is

$$\Lambda_* (x) = \mathbf{1}_{v>\rho_*} \frac{H_*^{n-1} (x)}{1 - \delta s \left(1 - H_*^{n-1} (x)\right)} \quad \forall x > \rho_*. \tag{4.14}$$

It indicates market competitiveness or buyer-friendliness of the market as we have mentioned.

In the next section, we partially use the change in the equilibrium sequential market to explore effects of the changes in different parameters of the model - the survival rate, the buyer-seller ratio, and the discount factor, on the equilibrium reserve price, stationary value distribution, buyer utility, seller revenue, and sale efficiency.

4.5 Comparative Statics

In this section, we investigate how the equilibrium is affected by changes in the market environment manifested in three parameters of the model - the buyer survival rate, the buyer-seller ratio, and the discount factor. Although there are only three parameters (apart from the newborn value distribution) in the model, each of them represents different characteristics of the market which we shall elaborate on.

We consider the change in the equilibrium from three perspectives: the sellers', the buyers', and the social planner's. It should not be a surprise that the sellers and the buyers always exhibit conflict of interests, but it is not directly obvious that the social planner's welfare aligns with the sellers'. In general, the changes that benefit the buyers and harm the sellers include prolonging buyer expected age, decreasing competition, and making them more patient. However, from the social efficiency point of view, benefiting the buyer worsens the allocative efficiency as the probability of sale decreases.

A seller's expected revenue from her optimally chosen auction $\mathrm{A}\left(\rho_*\right)$ is

$$r_* = \int_{\rho_*}^{1} \left[w_*\left(v\right) - \eta\left(H_*\left(v\right)\right) w_*'\left(v\right)\right] dH_*^n\left(v\right). \tag{4.15}$$

We need to characterize the changes in the optimal auction, the stationary buyer value distribution, and the buyer's WTP. In addition, a value v buyer's total discounted expected utility is

$$\underline{u}_*\left(v\right) = \mathbf{1}_{v \geq \rho_*} \int_{\rho_*}^{v} \frac{H_*^{n-1}\left(z\right)}{1 - s\delta\left(1 - H_*^{n-1}\left(z\right)\right)} dz. \tag{4.16}$$

There are different types of efficiencies associated with a sale mechanism. When the good is transferred from the seller who values the good at zero to any of the buyers, we say the transaction is **sale efficient**. When the sale occurs, and it is transferred to the buyer with the highest value, then we say the sale is **allocative efficient**. A zero reserve price second price auction is both sale efficient and allocative efficient as the good always ends up in the hand of the buyer with the highest valuation. A positive reserve price auction is not always sale efficient but is allocative efficient as the buyers' competition results in the allocative efficiency. However, an optimal posted price is neither sale nor allocative efficient. The significant posted price results in more ex-post sale inefficient allocation than the optimal auction because the optimal reserve price is lower than the optimal posted price, and it also results in allocative inefficiency as the good does not necessarily end in the hand of the buyer with the highest value.

The total social welfare is the discounted sum of the buyers and the sellers across all periods, but with a little careful consideration reveals that the equilibrium probability of sale is the key indicator of social efficiency. Since quasi-efficient auctions, the mechanisms that assign the good to the highest valued agent if not withheld by the seller, are run, and buyers and sellers divide the profit, transaction occurring is always preferred to otherwise as the sellers holding onto the good being the most socially inefficient allocation. The social efficiency monotonically increases in sale probability.

The platform designer's profit closely ties in with sale and allocative efficiency. If his profit is proportional to the total surplus he generates, given that a mechanism is allocative efficient as it is the case in an auction, higher sale efficiency results in higher total welfare. The idea that platform designer's maximum profit is proportional to the total welfare is supported by Oi (1971) and Armstrong (1999). If instead the designer charges a participation fee for new agents who arrive in the market (shown to be a revenue maximizing fee structure in Bodoh-Creed (2012)), he wishes that more new agents arrive, which is realized when equilibrium probability of sale increases.

Therefore, in order to consider the changes in the three perspectives, we solve the comparative statics of these variables - the seller's chosen equilibrium reserve price, the seller's utility; the stationary buyer value distribution, the buyer's utility; and the probability of sale. We summarize these results in the next three Propositions. All of the changes have definite signs and monotonic effects over the range as long as an additional convexity assumption on the newborn value distribution is imposed in addition to the existing one. It further restricts Assumption 3 of convex value distribution that guarantees the existence of equilibria, but not by much: the uniform distribution and any distribution with CDF $F(v) = v^k, k \geq 1$ still satisfies the assumption.

Assumption 5. *The newborn value distribution satisfies that $vf(v)/F(v)$ is increasing for $v < \rho^*_{mon}$.*

First, let us look at the change in the survival rate. These variations in the survival rate may reflect elasticities of demand across different types of goods: people are more likely to keep searching for a cellphone until they get one, but people may quit searching for a novel if they cannot find it at a satisfactory price. Furthermore, it can also represent search friction in the market; higher survival rate means that the current buyers can easily find a perfect substitute in the existing market. Lastly, recall that the market size growth rate is inversely related to the survival rate, an increase in survival rate also corresponds to a slower market expansion or faster market contraction.

Suppose that the survival rate increases so that the good becomes more inelastically demanded, the consumer's search friction decreases, or the consumer base remains relatively more stable. When the sellers survive to the next period with higher probability, the equilibrium stationary value distribution will be depressed because lower type buyers crowd the market, preventing newcomers to enter. The equilibrium reserve price the sellers choose goes down as a result. The probability of sale, positively related to the equilibrium reserve price, decreases. The buyers' expected utilities increase as they can search longer and the sellers revenue decreases.

Proposition 20 (Survival Rate)**.** *When the survival rate increases, the equilibrium reserve price decreases, the equilibrium probability of sale decreases, the stationary value distribution shifts down first-order stochastically, and the buyer's utility increases, and the seller's revenue decreases.*

The proof relies on the two equations that pin down the equilibrium reserve price and the stationary value distribution at the reserve price, (4.8) and (4.13). Algebraic rearrangements of the differentiation by the Implicit Function Theorem with the continuity of the solution guaranteed specifies the change in the reserve price, and with Assumption 5, the sign is definite. The ensuing changes follow from the change in reserve price. The proofs of the following two Propositions follow the same method, with the last one particularly easy as the discount factor does not affect the equilibrium conditions.

Decrease in the buyer-seller ratio, reduction in relative number of buyers to sellers, indicates that relative demand decreases or relative supply increases. Similar to the increase in survival rate, if the buyer-seller ratio decreases, the buyers are better off and the sellers are worse off. What is not obvious is that, although the reserve price decreases, the probability of sale decreases as well - the demand plummets more than what the supply side can optimally adjust for.

Proposition 21 (Buyer-Seller Ratio)**.** *When the buyer-seller ratio increases, the equilibrium reserve price decreases, the equilibrium probability of sale decreases, the stationary value distribution shifts down first-order stochastically, the buyer's utility increases, and the seller's revenue decreases.*

Finally, let us examine the effects of change in discount factor. If the discount factor increases, the buyers become more patient, or the interval between the time periods decreases and buyers have more sequential opportunities in the imminent future. The stationary value distribution and the reserve price are not affected by the change in discount factor, as the sellers do not change their optimal mechanism, as equilibrium reserve price only depends on the survival rate and the buyer-seller ratio.

In particular, we need to distinguish the shrinking time interval interpretation of the market expansion and the market growth rate as mentioned in Section 4.2. When the interval shrinks, the market is being replicated more times, increasing measures of the old buyers, the new buyers and the sellers. However, increasing the market growth rate only increases the measure of newborn buyers and sellers, and in particular decreasing the relative proportion of the old buyers whose opportunities are worsened rather than improved.

Proposition 22 (Discount Factor)**.** *When the discount factor increases, the equilibrium reserve price, the equilibrium probability of sale, the stationary value distribution are not affected, the buyer's utility increases and the seller's revenue decreases.*

Overall, when the survival rate increases and/or the buyer-seller ratio decreases, the buyers' expected utilities increase as their WTPs are depressed. The revenue of the sellers decreases, as they face a buyer composition that is first-order stochastically worse (of course, as a result of the precedent sellers' responses to buyer-friendly market environment). However, social sale efficiency decreases in response, resulting in lower probability of transactions. When buyers become more patient, the market composition does not change, but the seller's revenue will be hit hard, as the buyers are more willing to wait for future chances.

4.6 Posted Prices and Their Market Effects

Despite of being sub-optimal, the posted prices are robust revenue preservers to sequential price competitions. We define the posted prices marginal revenue and utilize it to explain why posted prices are an increasingly attractive for sellers when the market becomes more buyer friendly.

Let's consider the widely used posted price mechanism and its relation with the reserve price auction. A posted price $\mathrm{P}(\phi)$ is the mechanism where a seller posts price $w_\Lambda(\phi)$ and the buyers who are willing to buy at the price enter the lottery to be picked as a winner with equal chances. The posted price mechanism is an incentive compatible direct-revelation mechanism: only the buyers with willingness-to-pay $w_\Lambda(\phi)$, or values above ϕ enter the lottery. The lottery seems unrealistic, but it is equivalent to the following natural selling process when the buyers arrive stochastically. The seller posts price $w_\Lambda(\phi)$ and the n buyers uniformly randomly arrive within a time interval. The first buyer with willingness-to-pay greater than $w_\Lambda(\phi)$ takes the item and pays the posted price.

From the mechanism, all the buyers with value v above the **posted type** ϕ have the same expected utility,

$$u_\Lambda(v|\mathrm{P}(\phi)) = \frac{1}{n}\frac{1-F^n(\phi)}{1-F(\phi)}(v - w_\Lambda(\phi)),$$

and the seller's expected revenue is

$$r(\mathrm{P}(\phi)) = (1-F^n(\phi))\, w_\Lambda(\phi).$$

Define the **posted price marginal revenue in the presence of sequential market** Λ to be

$$\mathrm{MR}^{\mathrm{P}}_\Lambda(v) = w_\Lambda(v) - \eta(F^n(v))\, w'_\Lambda(v). \tag{4.17}$$

then the revenue-maximizing posted price mechanism in the presence of sequential market Λ is $\mathrm{P}^* = \mathrm{P}(\phi^*)$ with $\mathrm{MR}^{\mathrm{P}}_\Lambda(\phi^*) = 0$.

The posted price and the reserve price auction have a lot of similarities in terms of the marginal revenue curve. Both reserve type and posted type are determined by equating the marginal revenue to zero, and the types calculated specify the set of willing buyers. Together, we call the reserve price auction and the posted prices critical type mechanisms.

Remark 2. A critical type mechanism (CTM) $M(\tau)$ is either a reserve price auction $\mathrm{A}(\rho)$ or a posted price $\mathrm{P}(\phi)$. Its revenue is

$$r(M(\tau)) = \int_{\tau}^{1} \mathrm{MR}_{\Lambda}^{M}(v)\, dF^{n}(v)$$

and the optimal CTM is $M(\tau^*)$ where $\mathrm{MR}_{\Lambda}^{M}(\tau^*) = 0$. The probability of sale is $1 - F^n(\tau)$.

$\mathrm{MR}^{\mathrm{A}}(v) \geq \mathrm{MR}^{\mathrm{P}}(v)$ for all v because $\eta(F^n(v)) > \eta(F(v))$[4]. Therefore, the auction MR curve always dominates the posted price MR curve and the optimal posted type ϕ_{Λ}^* is always greater than the optimal reserve type ρ_{Λ}^*. The difference in revenues between the auction and the posted price of the same critical type is positive, and we call it the auction premium over posted price, $\Delta(\tau) \equiv r(\mathrm{A}(\tau)) - r(\mathrm{P}(\tau))$.

Whenever the sequential market becomes more buyer-friendly, the monopolist's revenues from the optimal auction and the optimal posted price decreases. Previous papers, realizing the disadvantages of an auction facing sequential competition, show that in the limit when the market is infinitely competitive, an auction and a posted price are equivalent, generating the same revenue (Kultti, 1999; Rachmilevitch and Reisman, 2012).

Proposition 23. *When the sequential market becomes more buyer-friendly or buyers become more patient, the auction premium over posted price decreases for any critical type.*

Wang (1993) shows in a stochastic setting that auction is more advantageous than posted price when the MR is steeper, Proposition (23) shows that more buyer friendly sequential market flattens the competitive auction marginal revenue more so that the gain in revenue is reduced. In addition to listing a critical type (the price posted or the reserve price in the auction), an auction involves additional procedures such as sorting the bids, announcing the

[4] $\eta(F^n(v)) = \eta(F(v))\left(1 + F(v) + \cdots + F^{n-1}(v)\right) / \left(nF^{n-1}(v)\right)$ is greater than $\eta(F(v))$ and decreasing when $F(\cdot)$ has decreasing inverse hazard rate.

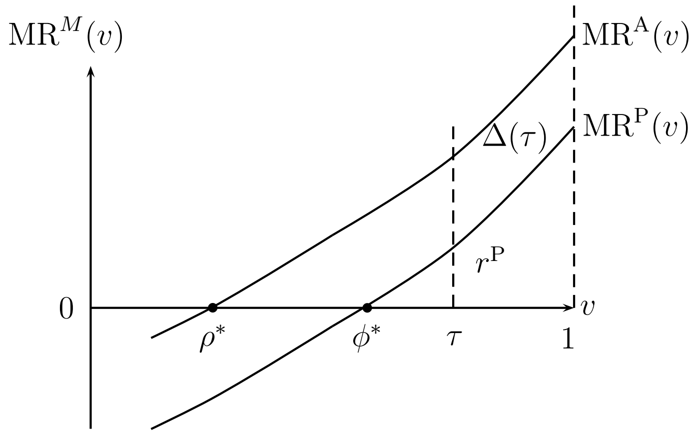

Figure 4.1: Critical type mechanisms.

winner and the payment calculated from the bids. The proof shows that the competitive marginal revenues of auction and posted price are closer to each other when the market becomes more buyer friendly or the buyers become more patient.

Proof of Proposition 23. The difference between the competitive marginal revenues is

$$\begin{aligned} \mathrm{MR}^{\mathrm{A}}_{\Lambda}(v) - \mathrm{MR}^{\mathrm{P}}_{\Lambda}(v) &= \left[w_{\Lambda}(v) - \eta\left(F^{n}(v)\right) w'_{\Lambda}(v)\right] - \left[w_{\Lambda}(v) - \eta\left(F^{n}(v)\right) w'_{\Lambda}(v)\right] \\ &= \left[\eta\left(F^{n}(v)\right) - \eta\left(F(v)\right)\right]\left(1 - \delta\Lambda(v)\right) \end{aligned}$$

The term in the square bracket is positive, so when $\tilde{\Lambda}(v) \geq \Lambda(v)$,

$$\mathrm{MR}^{\mathrm{A}}_{\tilde{\Lambda}}(v) - \mathrm{MR}^{\mathrm{P}}_{\tilde{\Lambda}}(v) \leq \mathrm{MR}^{\mathrm{A}}_{\Lambda}(v) - \mathrm{MR}^{\mathrm{P}}_{\Lambda}(v).$$

Also, when the discount factor becomes larger, the difference becomes strictly smaller as well as $\Lambda(v) > 0$. Take any critical type τ and the difference between the revenues in the presence of Λ is

$$r\left(\mathrm{A}(\tau)\right) - r\left(\mathrm{P}(\tau)\right) = \int_{\tau}^{1} \left[\mathrm{MR}^{\mathrm{A}}_{\Lambda}(v) - \mathrm{MR}^{\mathrm{P}}_{\Lambda}(v)\right] dF^{n}(v).$$

□

This result offers a possible explanation to the rapid demise of small auctions on eBay

(Einav et al., 2013). The paper argues that auctions are widely used in the early 2000s because the sellers are auction-fervent, but as time progresses with more entrance of general sellers, they do not have an inclination for posted prices. Another effect as the general population enters is increase in market liquidity so that buyers have more opportunities to purchase. When there are more better purchasing opportunities for the buyers in sequential periods, the revenue gain from running an auction, which declines as shown, may not cover the extra cost associated with displaying and storage.

It is also worth mentioning the change in optimal auction premium with respect to the number of buyers as it is not a monotonic relationship. When there is only one buyer, the seller essentially faces a bargaining problem: a buyer either accepts or rejects the price offer the seller proposes, so the optimal auction and the optimal posted price yield the same revenue. When there are several buyers, the auction yields the benefit that competition between buyers drives up the transaction price. However, when the number of buyers approaches infinity, the two revenues are equal again. There is probability 1 that a buyer has any arbitrarily high value, so full revenue of is guaranteed by posting a high price. What the sequential market limits is the maximum revenue any mechanism obtains, which is the willingness-to-pay of the highest value buyer, $w_\Lambda(1)$.

Figure 4.2 shows the ratio of the two revenues with respect to different number of buyers, when the buyers' values are drawn from uniform distribution $\left(r\left(\mathrm{P}_\Lambda^*\right)/r\left(\mathrm{A}_\Lambda^*\right),\ F(v)=v\right)$. 'X' shows the ratio when it is a pure monopoly setting, and 'O's shows the ratio when the sequential market is $\Lambda(x)=x$, i.e., each buyer expects to receive a price offer drawn from the uniform distribution. As the plot illustrates, the percentage of expected revenue the optimal posted price attains with respect to the optimal auction increases when there is a sequential market. For example, when there are five buyers, the seller can get 86% of the optimal revenue from a posted price as a monopoly but almost 90% of the optimal revenue when a sequential market is present. The absolute revenue difference exhibits a similar pattern - as

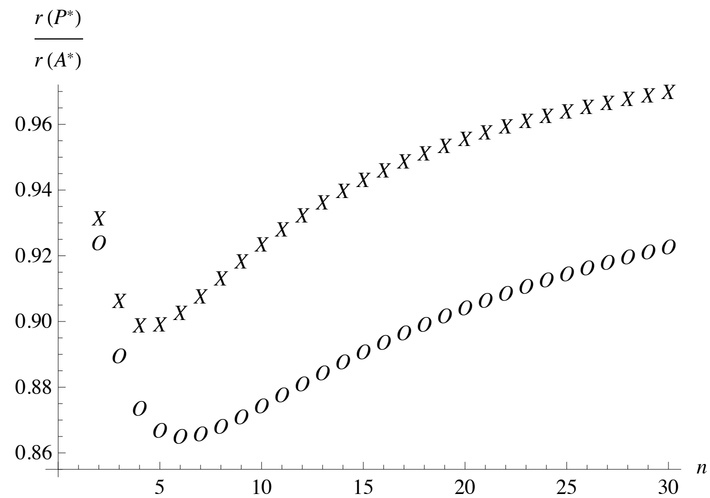

Figure 4.2: The ratio of the optimal auction and the optimal price revenues.

the market gets more buyer friendly, the optimal auction premium is likely to decrease.

If the seller is impatient and buyers arrive stochastically over time, posted price brings immediacy to transaction in the way that as soon as there is one willing buyer the transaction is realized and the payment is received, whereas the auction involves waiting longer. Therefore, when the buyer arrival rate is low, an impatient seller is more inclined to post a price instead of running the reserve price auction. Board and Skrzypacz (2014), for example, shows with continuos stochastic arrival of buyers, the optimal mechanism is to dynamically adjust prices with a reserve price auction in the end, so a posted price or an auction approximates the optimal revenue a seller. Skreta (2006) shows that posted price is optimal when the sellers cannot commit in a sequential setting.

We investigate market effects of such auction cost on equilibrium mechanism choice and market efficiency in the next section.

4.7 A Market with Heterogeneous Auction Costs

We have shown in the previous section that all the homogenous sellers run reserve price auctions in equilibrium. In this section, we investigate how seller's fixed cost of setting an auction alters his mechanism choice and what effects the switch to alternative mechanisms brings to the market.

Each seller chooses a critical type mechanism: $\mathcal{M} = \{\mathrm{A}(\cdot), \mathrm{P}(\cdot)\}$, but has heterogenous fixed cost associated with running an auction. The cost c is iid drawn from the cost distribution $G(\cdot)$ on support $[-1, 1]$ with positive PDF g. The expected profit $\pi(\mathrm{A}(\cdot))$ is the expected revenue $r(\mathrm{A}(\cdot))$ minus the cost c. The matching and market processes remain the same as in the previous setting. We consider both settings in which the sellers live for one period and two periods.

A SSSE $(M_*(\cdot), \sigma_*(\cdot,\cdot), H_*(\cdot), \mu_*(\cdot))$ in this setting only differs from Definition 6 by that each cost c seller chooses a possibly different mechanism $M_*(c)$. In equilibrium, there is a cutoff cost c_* such that any seller with cost $c > c_*$ uses posts price mechanism $\mathrm{P}(\phi_*)$ and any other with cost $c < c_*$ runs reserve price auction $\mathrm{A}(\rho_*)$. Cost c_* seller is indifferent between the two mechanisms because he yields the same profit,

$$\int_{\rho_*}^{1} \mathrm{MR}_*^{\mathrm{A}}(v)\, dH_*^n(v) - c_* = \int_{\phi_*}^{1} \mathrm{MR}_*^{\mathrm{P}}(v)\, dH_*^n(v).$$

The buyers with the highest values above exit slower than in the previous setting because all the participating buyers have the same probability of winning in a posted price. Furthermore, a posted price makes the sale probability smaller, resulting in more lower value buyers crowding the market, bringing down the stationary value distribution. In particular, such an equilibrium exists and is unique under the same assumptions as in the previous section. The derivations and characterizations of the SSSE are included in the Appendix.

We are particularly interested in the market effects of the auction costs and the posted price mechanisms as a consequence of the auction costs. The introduction of posted prices mechanisms reduces allocative efficiency possibly because sellers value immediacy of the sale to some buyer who arrives the earliest, not necessarily of the highest value. However, even though sale efficiency depreciates with the posted price mechanism compared to the auction mechanisms, the downward change in stationary value distribution brought by the posted prices actually improves sale efficiency of the equilibrium auction.

Proposition 24. *Suppose Assumption 5 holds. When there is auction cost, allocative inefficiency increases but change in sale efficiency is ambiguous.*

Although analytically it is ambiguous, under most circumstances, the sale efficiency decreases as the decrease in sale efficiency brought about by the switch to posted price dominates the gain in social efficiency in auctions. In the extreme case when every seller posts price, sale efficiency depreciates the most.

Note that the high cost sellers are the ones using inefficient, suboptimal posted price mechanisms. The sellers are gaining less from the posted price, because the sellers who use auctions get strictly higher profit $(r(\mathrm{A}(\rho_*)) - c > r(\mathrm{P}(\phi_*)))$. The highest value buyers' expected utilities are depressed too. If the sellers and buyers make decisions whether to enter the market based on ex-ante expected utilities, expected prevalence of posted prices mechanisms may deter their entrance, and possibly causes the market to collapse if there is a sufficiently competitive rival who may offer market of auctions. As a result, incentivizing the sellers to use auctions may be an attractive option. Facilitating buyer bids by allowing bidding bots, reducing fixed cost of auctions for sellers and buyers, and deterring posted prices may be options worth trying.

The theoretical predictions qualitatively match most qualitative results of empirical evidence. On the market level, the relative proportion of auctions to posted prices has rapidly decreased from over 95% in 2003 to only 25% in 2011 (Einav et al., 2013, Figure 1). However, share of revenues from auctions has not dropped as much, meaning that auctions generate relatively higher average revenues.

Empirical evidence suggests that the speed of convergence to SSSE is rather fast - three weeks in Ockenfels and Roth (2004). A deck of cards with each showing a most wanted terrorist was issued by the US military to its solders on April 11, 2003, but was sold on eBay and retail immediately (on day 2 - April 12, and on day 3 - April 13, respectively). The deck, for its novelty, origin, and rarity, sold for nearly \$70 in the first week, and mostly by auctions. However, the US Playing Cards Company released the identical copies of the cards for \$5.95 and got publicly known, the prices gradually dropped to the competitive prices in about three weeks and gradually higher proportion of decks were sold through posted prices.

4.8 Conclusion

The paper shows that reserve price auction remains a desirable choice for a seller facing uncertain demand and uncertain future competition. The reserve price, determined by a generalized marginal revenue curve, screens for buyers' willingnesses-to-pay but also indicates market competitiveness. The optimal posted price mechanism is shown to exhibit similar critical type determination procedure.

Although the optimal mechanism remains reserve price auction, sequential outside options reduce the desirability of an auction as it curtails some of the most important features of an auction. The advantage of running auction comes from the uniqueness of the item and the surplus extracted from the possible high valuations of the buyers, but the sequential opportunities make the current item dispensable, especially to buyers with high values for the item. A mechanism particularly immune to such turns in market conditions is posted price mechanism that does not have the transaction price dependent on unique willing buyers. Variants include dynamically adjusting prices and Buy-It-Now featured on eBay. However, posted prices are allocative inefficient and impose market externalities that affect sale efficiency and affect active buyer composition that lowers profit for future sellers.

We make some concluding remarks focusing on the current work's shortcomings and possible amendments for future works. An issue is the seller's commitment in case he does not sell, which is embodied by the assumption that we have maintained throughout the majority of the paper, sellers live only for one period and thus choose a static selling mechanism. When transaction costs of posting to intermediaries are high, when the items for sale have high depreciation rate or face intense competition from substitutable goods, a seller is better off using one-shot mechanisms. Reputation and rating systems also punish sellers for selling low quality goods for repeatedly many times. Furthermore, many goods that have substitutable competitions and upgrades have steep price drops in short period which essentially prevents a seller to choose a dynamically optimal pricing rule or repeated auctions. Zhang (2015b) shows that even if sellers can adjust their mechanisms periodically, it is going to be some predictable combination of auctions and posted prices. A dynamic model taking consideration of possible resale is of important interest.

Although we suggest several possibilities to the rise of auction cost - risk aversion, discount factor, fixed costs and idiosyncratic taste, the simple one-dimensional cost imposed is seemingly ad-hoc and does not tackle the more fundamental question *why* auction is less desirable to a rational agent who should supposedly only care about revenue. Works in dynamic mechanism design and revenue management can offer some insights.

A SSSE is proven to exist uniquely in the settings but we have been ignorant of the speed of convergence to such a stationary equilibrium behavior from an initial value distribution. If the speed of convergence to the SSSE is low (as it might be, suggested by numerical simulation of similar trading markets of Cho and Matsui (2012)), then it is important to investigate the adjustment on the equilibrium path to see how sellers switch from auctions to posted prices for example. Furthermore, we are agnostic about whether it is guaranteed that any change in the market environment results in global convergence to the new SSSE.

Furthermore, there is a line of empirical research that needs to be done, especially to the two latter dynamic markets. Equilibrium characterizations are complicated enough already that definitive comparative statics results were not obtainable because of complexity and non-monotonicity resulted from sellers' possible switch to alternative sub-efficient mechanisms. Numerical simulations and empirical works can be done to investigate different inefficiencies and externalities, to quantify the effects such as changes in revenues and probability of sale, and to test validity of assumptions made in the paper.

4.9 The Monopolist's Problem with Asymmetric Buyers

In this section, we solve the revenue-maximizing mechanism for n buyers with possibly asymmetric value distributions. Each buyer i has independent private value drawn distribution F_i on support $[\underline{v}_i, \overline{v}_i]$ such that $(1 - F_i(v))/f_i(v)$ is weakly decreasing. The sequential market is Λ, which is fixed throughout the section, so we do not include subscript. Let $\mathbf{v} = (v_1, \cdots, v_n)$ be the vector of values and $F(\mathbf{v}) = \prod_i F_i(v_i)$ be the value distribution.

A DSM $(P(\cdot), C(\cdot)) \equiv (P_i(\cdot), C_i(\cdot))_{i=1}^n$ consists of a collection of probability assignment functions and cost functions. The probability assignment functions take reports of the buyers to assign each buyer a probability of obtaining the item, with the properties that $0 \leq P_i(z_1, \cdots, z_n) \leq 1$ and $\sum_{i=1}^n P_i(z_1, \cdots, z_n) \leq 1$ for each buyer i. The cost function $C_i(z_1, \cdots, z_n)$ specifies the transfer from buyer i to the monopolist given all buyers' reports. Define the **expected probability assignment** and **expected cost function** as

$$\begin{aligned}
\overline{P}_i(z_i) &\equiv \int_{\prod_{j\neq i} z_j \in \prod_{j\neq i}[\underline{v}_j, \overline{v}_j]} P_i(z_1, \cdots, z_n) \prod_{j\neq i} dF_j(z_j), \\
\overline{C}_i(z_i) &\equiv \int_{\prod_{j\neq i} z_j \in \prod_{j\neq i}[\underline{v}_j, \overline{v}_j]} P_i(z_1, \cdots, z_n) \prod_{j\neq i} dF_j(z_j).
\end{aligned}$$

We want to restrict our attention to **incentive-compatible (IC)** mechanisms. Buyer i's expected utility of reporting z_i in the mechanism is the expected utility when he gets the object from the mechanism plus the expected utility if she does not get and waits until the next period,

$$u(z_i|v_i) = \overline{P}_i(z_i)\,v_i - \overline{C}_i(z_i) + \left(1 - \overline{P}_i(z_i)\right)\underline{u}(v_i),$$

Proposition 25. *$(P_i(\cdot), C_i(\cdot))_{i=1}^n$ is incentive-compatible if and only if for every buyer i,*

1. *$\overline{P}_i(v_i)$ is non-decreasing in v_i, and*
2. *$\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i)\,w(\underline{v}_i) + \overline{P}_i(v_i)\,w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(x)\,dw(x)$ for all $v_i \in [\underline{v}_i, \overline{v}_i]$.*

Proof of Proposition 25. First I show that if the mechanism is IC, then both conditions 1 and 2 hold. Define benefit of lying,

$$\psi_i(z|v_i) = u_i(z_i|v_i) - u_i(v_i|v_i).$$

IC implies that for any v_i, there is no benefit in lying, so $\psi_i(z_i|v_i) \le 0$ for all z_i, v_i, so

$$\begin{aligned}
0 &\ge \psi_i(z_i|v_i) + \psi_i(v_i|z_i) \\
&= \left[\overline{P}_i(z_i)\,w(v_i) - \overline{C}_i(z_i) + (v_i - w(v_i)) - \left(\overline{P}_i(v_i)\,w(v_i) - \overline{C}_i(v_i) + (v_i - w(v_i))\right)\right] - \\
&\quad \left[\overline{P}_i(v_i)\,w(z_i) - \overline{C}_i(v_i) + (z_i - w(z_i)) - \left(\overline{P}_i(z_i)\,w(z_i) - \overline{C}_i(z_i) + (z_i - w(z_i))\right)\right] \\
&= \left[\overline{P}_i(z_i)\,w(v_i) - \overline{C}_i(z_i) - \left(\overline{P}_i(v_i)\,w(v_i) - \overline{C}_i(v_i)\right)\right] - \\
&\quad \left[\overline{P}_i(v_i)\,w(z_i) - \overline{C}_i(v_i) - \left(\overline{P}_i(z_i)\,w(z_i) - \overline{C}_i(z_i)\right)\right] \\
&= \left(\overline{P}_i(z_i) - \overline{P}_i(v_i)\right)(w(v_i) - w(z_i))
\end{aligned}$$

Since $w(v_i)$ is non-decreasing in v_i, $\overline{P}_i(v_i)$ is non-decreasing in v_i. Furthermore, IC implies that $u_i(z_i|v_i)$ is maximized at $z_i = v_i$, then by envelope theorem,

$$\left.\frac{\partial u_i(z_i|v_i)}{\partial z_i}\right|_{z_i = v_i} = \overline{P}_i'(v_i)\,w(v_i) - \overline{C}_i'(v_i) = 0.$$

By fundamental theorem of calculus,

$$\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} \overline{P}_i'(v_i)\, w(v_i)\, dv_i$$

Integration by parts yields condition 2 as desired. The converse is shown directly by definition of incentive-compatibility:

$$\begin{aligned}\psi_i(z_i|v_i) &= \left[\overline{P}_i(z_i) w(v_i) - \overline{C}_i(z_i) + (v_i - w(v_i))\right] \\ &\quad - \left[\overline{P}_i(v_i) w(v_i) - \overline{C}_i(v_i) + (v_i - w(v_i))\right] \\ &= \left[\overline{P}_i(z_i) - \overline{P}_i(v_i)\right] w(v_i) + \overline{C}_i(v_i) - \overline{C}_i(z_i)\end{aligned}$$

Plugging in condition 2,

$$\begin{aligned}\psi_i(z_i|v_i) &= \left[\overline{P}_i(z_i) - \overline{P}_i(v_i)\right] w(v_i) + \overline{P}_i(v_i) w(v_i) - \int_{z_i}^{v_i} \overline{P}_i(x)\, dw(x) - \overline{P}_i(z_i) w(z_i) \\ &= \overline{P}_i(z_i)\left[w(v_i) - w(z_i)\right] - \int_{z_i}^{v_i} \overline{P}_i(x)\, dw(x) = \int_{z_i}^{v_i} \left(\overline{P}_i(z_i) - \overline{P}_i(x)\right) dw(x)\end{aligned}$$

Then by condition 1, the expression is non-positive. □

Because the expected probability assignment function pins down the expected cost function, if the expected cost function of the buyer of the lowest value is the same across two bidders and the expected probability assignment is the same, then the expected revenue is the same for the seller, and all buyers are indifferent between the incentive-compatible mechanisms the seller runs.

Corollary 1 (Buyer Indifference and Revenue Equivalence)**.** *If two incentive-compatible mechanisms have 1) the same expected probability assignment functions, and 2) the same expected costs for the buyer of the lowest value, then all the agents are indifferent between the two mechanisms, as they yield the same expected revenue, and the same expected payoffs for the buyers of the same value.*

Proof to Corollary 1. Denote the two mechanisms $M^I = \left(\overline{P}_i^I(\cdot), \overline{C}_i^I(\cdot)\right)_{i=1}^n$ and $M^{II} = \left(\overline{P}_i^{II}(\cdot), \overline{C}_i^{II}(\cdot)\right)_{i=1}^n$, respectively. In any IC mechanism $M = \left(\overline{P}_i(\cdot), \overline{C}_i(\cdot)\right)_{i=1}^n$, the expected cost function

$$\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i)\tilde{v}(\underline{v}_i) + \overline{P}_i(v_i)w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(x)\,dw(x)$$

Because $\overline{P}_i^I(\cdot) = \overline{P}_i^{II}(\cdot)$ for all i by condition 1 and $\overline{C}_i^I(\underline{v}_i) = \overline{C}_i^{II}(\underline{v}_i)$ by condition 2, $\overline{C}_i^I(v_i) = \overline{C}_i^{II}(v_i)$ for all v_i for all i. Expected utility of v_i in mechanism M is

$$u_i^M(v_i|v_i) \;\; = \;\; \overline{P}_i(v_i)w(v_i) - \overline{C}_i(v_i) + v_i - w(v_i)$$

$u_i^{M^I}(v_i|v_i) = u_i^{M^{II}}(v_i|v_i)$ for all v_i for i. Revenue is determined by

$$r(M) = \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} \overline{C}^M(v_i)\,dF_i(v_i)$$

Then as shown that $\overline{C}_i^I(v_i) = \overline{C}_i^{II}(v_i)\ \forall v_i\ \forall i$, $r\left(M^I\right) = r\left(M^{II}\right)$. □

The seller maximizes expected revenue subject to the incentive compatibility and individual rationality constraints.

Definition 8. A revenue-maximizing IC, IR DSM is $M = (P_i(\cdot), C_i(\cdot))_{i=1}^n$ that maximizes

$$r \;\; = \;\; \sum_{i=1}^n \int_{\underline{v}_i}^{\overline{v}_i} \overline{C}_i(v_i)\,dF_i(v_i)$$

subject to $\forall i$: $\tilde{u}_\Lambda(v_i|v_i) \geq \tilde{u}_\Lambda(z_i|v_i)\ \forall z_i \neq v_i$ and $\tilde{u}_\Lambda(v_i|v_i) \geq 0\ \forall v_i$.

Proposition 26 (The Optimal Mechanism)**.** *The revenue-maximizing IC, IR DSM* $(P_i^*(\cdot),$ $C_i^*(\cdot))_{i=1}^n$ *is that* $P_i^*(\mathbf{v}) = 1$ *if* $\tilde{v}(v_i) - \frac{1-F_i(v_i)}{f_i(v_i)}w'(v_i) > \max\left\{\tilde{v}(v) - \frac{1-F_j(v_j)}{F_j(v_j)}w'(v_i), 0\right\}$ *and* $= 0$ *otherwise, and*

$$\overline{C}_i^*(v_i) = \overline{C}_i^*(\underline{v}_i) - \overline{P}_i^*(\underline{v}_i)\tilde{v}(\underline{v}_i) + \overline{P}_i^*(v_i)\tilde{v}(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i^*(z)\,d\tilde{v}(z)$$

The several key steps to the proof are i) characterizing the individual rationality constraint, ii) exchanging integrals, and iii) expanding expected probability assignment functions to obtain a weighted average of probability assignment functions.

Proof of Proposition 26. The constraints are

1. IC1: $\overline{P}_i(v_i) \geq \overline{P}_i(z_i)\ \forall v_i \geq z_i$,

2. IC2: $\overline{C}_i(v_i) = \overline{C}_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} \overline{P}'_i(v_i)\, w(v_i)\, dv_i\ \forall v_i$, and

3. IR: $u_i(v_i|v_i) \geq \underline{u}_i(v_i)$.

In particular, The IR implies that $\overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i)\, w(\underline{v}_i) \leq 0$.

$$r = \sum_{i=1}^{n} \int_{\underline{v}_i}^{\overline{v}_i} \left[\overline{P}_i(v_i)\, w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(z)\, dw(z) \right] dF_i(v_i) + \sum_{i=1}^{n} \left[\overline{C}_i(\underline{v}_i) - \overline{P}_i(\underline{v}_i)\, w(\underline{v}_i) \right]$$

where

$$\begin{aligned}
& \int_{\underline{v}_i}^{\overline{v}_i} \left[\overline{P}_i(v_i)\, w(v_i) - \int_{\underline{v}_i}^{v_i} \overline{P}_i(x)\, dw(x) \right] dF_i(v_i) \\
= \quad & \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(v_i)\, w(v_i)\, f_i(v_i)\, dv_i - \int_{\underline{v}_i}^{\overline{v}_i} \int_{x}^{\overline{v}_i} \overline{P}_i(x)\, w'(x)\, f_i(v_i)\, dv_i dx \\
= \quad & \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(v_i)\, w(v_i)\, f_i(v_i)\, dv_i - \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(x)\, w'(x)\, (1 - F_i(x))\, dx \\
= \quad & \int_{\underline{v}_i}^{\overline{v}_i} \overline{P}_i(v_i) \left[w(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} w'(v_i) \right] f_i(v_i)\, dv_i \\
= \quad & \int_{\underline{v}_1}^{\overline{v}_1} \cdots \int_{\underline{v}_n}^{\overline{v}_n} P_i(\mathbf{v}) \left[w(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} w'(v_i) \right] dF(\mathbf{v})
\end{aligned}$$

Therefore,

$$r \quad \leq \quad \int_{\underline{v}_1}^{\overline{v}_1} \cdots \int_{\underline{v}_n}^{\overline{v}_n} \sum_{i=1}^{n} P_i(\mathbf{v}) \left[w(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} w'(v_i) \right] dF(\mathbf{v}),$$

and any mechanism that implements the specified probability assignment function will achieve the upper bound. □

4.10 Proofs

Proof of Proposition 16. It follows as a corollary of Proposition 26 when all the value distributions are identically F. When the WTP function $w(\cdot)$ is not continuously differentiable, the optimal reserve type ρ^* satisfies that $\mathrm{MR}^{\mathrm{A}}\left(\rho_+^*\right) \geq 0$ and $\mathrm{MR}^{\mathrm{A}}\left(\rho_-^*\right) \leq 0$. The existence and uniqueness of ρ^* are guaranteed by the monotonicity of the auction marginal revenue curve. □

Proof of Proposition 17. The optimal reserve type is determined by

$$w_\Lambda\left(\rho_\Lambda^*\right)/w_\Lambda'\left(\rho_\Lambda^*\right) = \eta\left(F\left(\rho_\Lambda^*\right)\right)$$

The LHS is increasing and the RHS is decreasing. If we show that LHS is greater than ρ_Λ^*, then $\rho_\Lambda^* < \rho_{\mathrm{mon}}^*$. $w_\Lambda(v)/w_\Lambda'(v) \geq v$ if

$$\begin{aligned} w_\Lambda(v) - v w_\Lambda'(v) &= v - \delta\int_{\underline{x}}^{v}\Lambda(x)\,dx - v\left(1 - \delta\Lambda(v)\right) \\ &= v\delta\Lambda(v) - \delta\int_{\underline{x}}^{v}\Lambda(x)\,dx \geq 0 \end{aligned}$$

as $\Lambda(x) \leq \Lambda(v)$ for all $x \leq v$. □

Proof of Proposition 18. The derivation modifies Jehle and Reny (2011). Value v buyer's utility bidding according to the strictly increasing bidding function $\sigma(z)$ is

$$u\left(v, \sigma(z), \sigma_{-i}(\cdot)\,|\mathrm{A}(\rho)\right) = F^{n-1}(z)\left(v - \sigma(z)\right) + l(z)\,\delta\underline{u}_\Lambda(v).$$

where $l(z) = 1 - \mathbf{1}_{v\geq\rho}F^{n-1}(z)$ is the expected probability of losing by reporting z. Rearrange,

$$u\left(v, \sigma(z), \sigma_{-i}(\cdot)\,|\mathrm{A}(\rho)\right) = F^{n-1}(z)\left(w_\Lambda(v) - \sigma(z)\right) + \delta\underline{u}_\Lambda(v) \tag{4.18}$$

Differentiate (4.18) with respect to z and the buyer optimality condition requires that it is zero at $z = v$. Coupled with the boundary condition that $\sigma(\rho) = w_\Lambda(\rho)$, the equilibrium bidding function is obtained. The candidate bidding function is indeed the equilibrium as the utility is maximized (rather than minimized) at $z = v$. The equilibrium payoffs are obtained by replugging in the equilibrium bidding function, and revenue is obtained by the MR curve by Riley and Samuelson (1981). □

Proof of Proposition 19. We show first that there exists a unique solution of ρ_* to (4.8) and (4.13), and it is the only candidate equilibrium. It is then shown to be the unique optimal reserve type that a seller chooses in the equilibrium. Coupled with the stationary distributions determined by (4.11) and (4.12), it constitutes an equilibrium.

First we show the existence and uniqueness of ρ_* and $H_*(\rho_*)$ for the system of equations, (4.8) and (4.13). If every seller chooses the same reserve type ρ auction (not necessarily optimal), the stationary value distribution resulted from it is characterized by (4.12) (substitute ρ for ρ_*). In particular, $x \equiv H_*(\rho)$ is determined by (4.13),

$$x / \left[1 + \frac{s}{1-s} \frac{1}{n} (1 - x^n) \right] = F(\rho).$$

LHS is monotonically strictly increasing in x, ranging from 0 (when $x = 0$) to 1 (when $x = 1$). Since RHS is a fixed number between 0 and 1 for any ρ, there is solution $x(\rho) = H_*(\rho)$ for each ρ, and is strictly increasing in ρ.

In order to show that (4.8) has a solution, it is sufficient to show that LHS is increasing in ρ, and it holds when $1/\eta(H_*(\rho))$ is increasing,

$$\frac{h_*(\rho)}{1 - H_*(\rho)} = \frac{H_*(\rho)}{1 - H_*(\rho)} \frac{f(\rho)}{F(\rho)} = f(\rho) \left[\frac{1}{1 - H_*(\rho)} + \frac{s}{1-s} \frac{1}{n} \frac{1 - {H_*}^n(\rho)}{1 - H_*(\rho)} \right]$$

where both equalities follow from (4.13). f is increasing by Assumption (3) and the second term is increasing in ρ because $H_*(\rho)$ is increasing. Overall, LHS of (4.8) continuously and monotonically increases from $-1/f(0)$ to 1 as ρ increases from 0 to 1. Therefore, the solution to the system of equations, ρ_*, is unique.

In fact, it is the only candidate equilibrium. Any ρ can be the symmetric equilibrium reserve type but the stationary distribution from such choice of ρ is pinned down by solution $H_*(\rho) = x(\rho)$. Only the pairs $(\rho, x(\rho))$ satisfy stationarity condition and only one pair is candidate equilibrium by the seller's optimality condition that determines the reserve price. Finally, it is sufficient to show that under $H_*(\rho_*)$, ρ_* is indeed the unique optimal reserve price.

ρ_* is optimal if $\mathrm{MR}_*^{\mathrm{A}}(v)$ is increasing, so it is sufficient to show that $h_*(v)/[(1 - H_*(v))w_*'(v)]$ is increasing.

$$\frac{d\log\left[h_*(v)/\left[(1-H_*(v))\,w_*'(v)\right]\right]}{dv} = \frac{h_*'(v)}{h_*(v)} + \frac{h_*(v)}{1-H_*(v)} - \frac{w_*''(v)}{w_*'(v)}.$$

By (4.2),

$$(1-sl_*(v))\,h_*(v) = (1-sl_*)\,f(v) \Rightarrow (1-sl_*(v))\,h_*'(v) - sl_*'(v)\,h_*(v) = (1-sl_*)\,f'(v)$$

so

$$\frac{h_*'(v)}{h_*(v)} = \frac{f'(v)}{f(v)} + \frac{sl_*'(v)}{1-sl_*(v)}$$

By differentiating (4.20),

$$w_*''(v) = w_*'(v)\,\frac{\delta sl_*'(v)}{1-\delta sl_*(v)}.$$

Pulling together, we get

$$\begin{aligned} & \frac{f'(v)}{f(v)} + \frac{sl_*'(v)}{1-sl_*(v)} + \frac{h_*(v)}{1-H_*(v)} - \frac{\delta sl_*'(v)}{1-\delta sl_*(v)} \\ = \; & \frac{f'(v)}{f(v)} + \frac{h_*(v)}{1-H_*(v)} + \frac{(1-\delta)\,sl_*'(v)}{(1-sl_*(v))\,(1-\delta sl_*(v))} \\ = \; & \frac{f'(v)}{f(v)} + h_*(v)\left[\frac{1}{1-H_*(v)} - \frac{(1-\delta)\,s}{(1-sl_*(v))\,(1-\delta sl_*(v))}\,(n-1)\,H_*^{n-2}(v)\right] \end{aligned}$$

Note that $\frac{1}{1-H_*(v)} \geq \sum_{j=0}^{n-2} H_*^j(v) \geq (n-1)\,H_*^{n-2}(v)$, the term in the bracket is greater than

0 if

$$\frac{(1-\delta) s}{(1-s l_*(v))(1-\delta s l_*(v))} \leq 1$$

and the LHS is bounded by $\frac{(1-\delta)s}{(1-s)(1-\delta s)}$ when $l_*(v) = 1$. The numerator is smaller than the (positive) denominator when Assumption (4) holds. Coupled with Assumption (3) that guarantees f' to be positive, the auction marginal revenue curve is strictly increasing, so ρ_* is the only solution to $\mathrm{MR}_*^{\mathrm{A}}(\rho) = 0$, thus the only equilibrium. □

Calculations of Sequential Market, WTP, Equilibrium Buyer Utility and Seller Revenue Because the equilibrium behavior and buyer composition are stationary, value $v > \rho_*$ buyer's total discounted expected payoff is the same as her continuation payoff, which is the same as the equilibrium expected payoff in $\mathrm{A}(\rho_*)$, which by (4.6) is,

$$\underline{u}_*(v) = u(v|\mathrm{A}(\rho_*)) = \int_{\rho_*}^{v} H_*^{n-1}(z)\, w_*'(z)\, dz + s\delta \underline{u}_*(v). \tag{4.19}$$

Her WTP is her value net her present value of expected continuation payoff, depressed by her survival rate and discount factor,

$$w_*(v) = v - s\delta \underline{u}_*(v) = v - \frac{s\delta}{1-s\delta} \int_{\rho_*}^{v} H_*^{n-1}(z)\, w_*'(z)\, dz \tag{4.20}$$

where (4.20) is from plugging in (4.19). Differentiate both sides of (4.20) and rearrange, we get that

$$w_*'(v) = \frac{1-s\delta}{1-s\delta\left(1-H_*^{n-1}(v)\right)} = \frac{1-s\delta}{1-s\delta l_*(v)}.$$

In equilibrium, value v buyer's equilibrium WTP is

$$w_*(v) = v - \mathbf{1}_{v \geq \rho_*} \int_{\rho_*}^{v} \frac{s\delta H_*^{n-1}(z)}{1-s\delta+s\delta H_*^{n-1}(z)} dz. \tag{4.21}$$

Proof of Proposition 20. The equilibrium ρ_* and $H_*(\rho_*)$ simultaneously satisfy

$$H_*(\rho_*) \equiv \left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] F(\rho_*) \tag{4.22}$$

$$\rho_* \cdot h_* (\rho_*) + H_* (\rho_*) \;\; \equiv \;\; 1 \tag{4.23}$$

Plugging in the equality $H_* (\rho_*) / h_* (\rho_*) = F (\rho_*) / f (\rho_*)$ to (4.23),

$$H_* (\rho_*) = F (\rho_*) / (F (\rho_*) + \rho_* f (\rho_*)) \tag{4.24}$$

and plug that into (4.22),

$$\frac{1}{F (\rho_*) + \rho_* f (\rho_*)} = 1 + \frac{s}{1-s}\frac{1}{n}\left(1 - \left(\frac{F (\rho_*)}{F (\rho_*) + \rho_* f (\rho_*)}\right)^n\right) \tag{4.25}$$

This holds for all s, n at $\rho_* (s, n)$, the equilibrium reserve price at the respective environment. By Implicit Function Theorem on (4.24),

$$\frac{dH_* (\rho_*)}{ds} = \left(\frac{1}{1 + \frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)' \frac{d\rho_*}{ds} \tag{4.26}$$

has the opposite sign as $d\rho_*/ds$, because the sign of the first term is negative by Assumption 5. By Implicit Function Theorem again, differentiate (4.25) with respect to s,

$$\left(\frac{1}{F (\rho_*) + \rho_* f (\rho_*)}\right)' \frac{d\rho_*}{ds} \;\; = \;\; \left(\frac{s}{1-s}\right)' \frac{1}{n} (1 - H_*^n (\rho_*)) - \frac{s}{1-s} H_*^{n-1} (\rho_*) \frac{dH_* (\rho_*)}{ds}$$

Plug (4.26) in and rearrange,

$$\left[\left(\frac{1}{F (\rho_*) + \rho_* f (\rho_*)}\right)' + \frac{s}{1-s} H_*^{n-1} (\rho_*) \left(\frac{1}{1 + \frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'\right] \frac{d\rho_*}{ds}$$
$$= \;\; \left(\frac{s}{1-s}\right)' \frac{1}{n} (1 - H_*^n (\rho_*))$$

Since $\left(\frac{s}{1-s}\right)' > 0$, the sign of $\frac{d\rho_*}{ds}$ is the same as

$$\left(\frac{1}{F (\rho_*) + \rho_* f (\rho_*)}\right)' + \frac{s}{1-s} H_*^{n-1} (\rho_*) \left(\frac{1}{1 + \frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'.$$

The first term is negative because $f' (v) \geq 0$ and the second term is negative by Assumption 5. Therefore, $d\rho_*/ds < 0$: the sign is strict because $\rho_* f (\rho_*)$ is strictly increasing.

The change in probability of sale with respect to higher survival rate has opposite sign as that of $dH_*(\rho_*)/ds$ which has opposite sign as $d\rho_*/ds$, so it is negative.

By (4.12), for all $v < \rho_*$,

$$H_*(v) = F(v)\left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right].$$

Since the term in the bracket equals $1/\left(F(\rho_*) + \rho_* f(\rho_*)\right)$, it increases as s increases:

$$\frac{d\left[1/\left(F(\rho_*) + \rho_* f(\rho_*)\right)\right]}{ds} = \left(\frac{1}{F(\rho_*) + \rho_* f(\rho_*)}\right)' \frac{d\rho_*}{ds} > 0,$$

as both multiplicands are negative. For all $v \geq \rho_*$,

$$H_*(v) = F(v)\left[1 + \frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right)\right] - \frac{s}{1-s}\frac{1}{n}\left(H_*^n(v) - H_*^n(\rho_*)\right)$$

Rearrange,

$$H_*(v) + \frac{s}{1-s}\frac{1}{n}H_*^n(v) = F(v) + F(v)\frac{s}{1-s}\frac{1}{n}\left(1 - H_*^n(\rho_*)\right) + \frac{s}{1-s}\frac{1}{n}H_*^n(\rho_*) \tag{4.27}$$

Differentiate with respect to s and let $x \equiv H_*(v)$, the LHS is

$$\frac{dx}{ds} + \frac{s}{1-s}x^{n-1}\frac{dx}{ds} + \left(\frac{s}{1-s}\right)'\frac{1}{n}H_*^n(v),$$

and the RHS is

$$F(v)\left(\frac{s}{1-s}\right)'\frac{1}{n}\left(1 - H_*^n(\rho_*)\right) + \left(\frac{s}{1-s}\right)'\frac{1}{n}H_*^n(\rho_*) + \left(1 - F(v)\right)\frac{s}{1-s}\frac{1}{n}nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{ds}$$

Rearrange, $\left(1 + \frac{s}{1-s}x^{n-1}\right)\frac{dx}{ds}$ equals

$$\left[F(v)\frac{1}{n}\left(1 - H_*^n(\rho_*)\right) - \frac{1}{n}\left(H_*^n(v) - H_*^n(\rho_*)\right)\right]\left(\frac{s}{1-s}\right)' + \left(1 - F(v)\right)\frac{s}{1-s}\frac{1}{n}nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{ds}$$

To show that $dx/ds \geq 0$, it suffices to show that the first term in the bracket is positive, because $\frac{dH_*(\rho_*)}{ds} \geq 0$ is already shown. By (4.12), the term equals

$$[H_*(v) - F(v)] / \left(\frac{s}{1-s}\right) \geq 0$$

as the newborn distribution always first order stochastically dominates the stationary value distribution in equilibrium.

The buyer's utility then increases as the sequential market becomes more buyer-friendly. For (4.16), the integrand increases as $H_*(v)$ increases for all $v > \rho_*$, s increases, and ρ_* decreases. On the other hand, the seller's revenue decreases because the willingnesses to pay of the buyers all decrease, and the buyer stationary value distribution first stochastically increases, resulting in less equilibrium probability of sale. □

Proof of Proposition 21. The comparative statics results still derive from the equilibrium conditions of (4.22) and (4.23). Differentiate (4.25) with respect to n, the LHS is

$$\left(\frac{1}{F(\rho_*) + \rho_* f(\rho_*)}\right)' \frac{d\rho_*}{dn}$$

and the RHS becomes

$$\begin{aligned}
& \frac{s}{1-s}\left(-\frac{1}{n^2}\right)(1 - H_*^n(\rho_*)) + \frac{s}{1-s}\left(-nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{dn} - H_*^n(\rho_*)\log H_*(\rho_*)\right) \\
= \; & -\frac{s}{1-s}nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{dn} - \frac{s}{1-s}\frac{1}{n^2}\left[1 - H_*^n(\rho_*) + H_*^n(\rho_*)\log H_*^n(\rho_*)\right]
\end{aligned}$$

Equating the two sides and rearrange,

$$\begin{aligned}
& -\left[\left(\frac{1}{F(\rho_*) + \rho_* f(\rho_*)}\right)' + \frac{s}{1-s}nH_*^{n-1}(\rho_*)\left(\frac{1}{1 + \frac{\rho_* f(\rho_*)}{F(\rho_*)}}\right)'\right]\frac{d\rho_*}{dn} \\
= \; & \frac{s}{1-s}\frac{1}{n^2}\left[1 - H_*^n(\rho_*) + H_*^n(\rho_*)\log H_*^n(\rho_*)\right]
\end{aligned}$$

The two terms in the bracket in the LHS are both negative as shown in the previous proof (first by Assumption 3 and second by Assumption 5). Therefore $\frac{d\rho_*}{dn}$ has the same sign as that of RHS. Since $H_*^n(\rho_*) < 1$,

$$H_*^n(\rho_*)\left(1-\log H_*^n(\rho_*)\right)-1=(1+x)/\exp(x)-1<0$$

for $x=-\log H_*^n(\rho_*)>0$ as $\exp(x)>1+x$ by Taylor expansion. Differentiation of (4.22) yields the same result as with s, so it is (4.26) with n replacing s, so $dH_*(\rho_*)/dn < 0$. Furthermore,

$$\begin{aligned} d\left(1-H_*^n(\rho_*)\right)/dn &= -H_*^n(\rho_*)\log H_*(\rho_*)-nH_*^{n-1}(\rho_*)\frac{dH_*(\rho_*)}{dn} \\ &= -H_*^{n-1}(\rho_*)\left(H_*(\rho_*)\log H_*(\rho_*)+n\frac{dH_*(\rho_*)}{dn}\right)>0. \end{aligned} \quad (4.28)$$

In summary thus far, $d\rho_*/dn>0$, $dH_*(\rho_*)/dn<0$, and $d\left(1-H_*^n(\rho_*)\right)/dn>0$.

Next, we show that $dH_*(v)/dn<0$ for all v. First, $\left[1+\frac{s}{1-s}\frac{1}{n}\left(1-H_*^n(\rho_*)\right)\right]$ is decreasing because its change equals

$$\left(\frac{1}{F(\rho_*)+\rho_* f(\rho_*)}\right)'\frac{d\rho_*}{dn}$$

which is negative, as the first term is negative and the second term positive. Therefore, $H_*(v)$ decreases for all $v<\rho_*$ as n increases. Next, differentiate (4.27) with respect to n, the LHS equals ($x\equiv H_*(v)$),

$$\left[1+\frac{s}{1-s}H_*^{n-1}(v)\right]\frac{dx}{dn}-\frac{s}{1-s}\frac{1}{n^2}\left(H_*^n(v)-H_*^n(v)\log H_*^n(v)\right).$$

The RHS equals the derivative of

$$\frac{s}{1-s}\frac{1}{n}-(1-F(v))\left[1-\frac{s}{1-s}\frac{1}{n}\left(1-H_*^n(\rho_*)\right)\right]$$

which is

$$\frac{s}{1-s}\left[-\frac{1}{n^2}+(1-F(v))\,d\left(\frac{1}{n}\left(1-H_*^n(\rho_*)\right)\right)/dn\right]$$

Rearrange the terms, then equals

$$\left[1+\frac{s}{1-s}H_*^{n-1}(v)\right]\frac{dx}{dn} = -\frac{s}{1-s}\frac{1}{n^2}\left[1-H_*^n(v)+H_*^n(v)\log H_*^n(v)\right] + \frac{s}{1-s}(1-F(v))\,d\left(\frac{1}{n}(1-H_*^n(\rho_*))\right)/dn<0$$

where the first term being negative follows from (4.28) and the second term being negative follows from $\left[1+\frac{s}{1-s}\frac{1}{n}(1-H_*^n(\rho_*))\right]$ decreasing in n.

Finally, the buyer's utility decreases and the seller's revenue increases because the WTP increases for all buyers of different values. □

Proof of Proposition 22. Change in δ does not affect the equilibrium reserve price, so the equilibrium probability of sale and stationary value distribution are not affected either. The buyer's utility increases because the sequential market becomes more buyer-friendly: directly by (4.16), increase in δ increases the integral. The seller's revenue decreases as each buyer's WTP decreases and the buyer composition does not change. The reduction in WTP is by (4.21),

$$w_*(v)=v-\mathbf{1}_{v\geq\rho_*}\int_{\rho_*}^{v}\frac{H_*^{n-1}(z)}{\left(\frac{1}{\delta s}-1\right)+H_*^{n-1}(z)}dz.$$

□

In the equilibrium, for a cost c seller facing n symmetric buyers whose WTP is determined by $w_*(\cdot)$, the profit-maximizing auction is $\mathrm{A}(\rho_*)$ where ρ_* is determined by

$$\rho_*-\frac{1-H_*(\rho_*)}{h_*(\rho_*)}=0$$

and the expected profit from it is $\pi(\mathrm{A}(\rho_*))=r(\mathrm{A}(\rho_*))-c$, where

$$r(\mathrm{A}(\rho_*))=\int_{\rho_*}^{1}\left[w_*(v)-\frac{1-H_*(v)}{h_*(v)}w_*'(v)\right]dH_*^n(v).$$

The profit-maximizing posted price is $\mathrm{P}(\phi_*)$ where ϕ_* is determined by

$$w_*(\phi_*) - \frac{1 - H_*^n(\phi_*)}{(H_*^n(\phi_*))'} w_*'(\phi_*) \quad = \quad 0$$

and the expected profit and revenue from it is

$$\pi(\mathrm{P}(\phi_*)) = r(\mathrm{P}(\phi_*)) = (1 - H_*^n(\phi_*)) w_*(\phi_*).$$

Since a seller can only run an auction or post a price, he essentially makes a choice between $\mathrm{A}(\rho_*)$ and $\mathrm{P}(\phi_*)$, and he will choose the auction if and only if the auction generates higher expected profit, or equivalent, the equilibrium auction premium is greater than the cost,

$$\pi(\mathrm{A}(\rho_*)) \geq \pi(\mathrm{P}(\phi_*)) \Leftrightarrow \Delta_* = r(\mathrm{A}(\rho_*)) - r(\mathrm{P}(\phi_*)) \geq c,$$

and posts the optimal price otherwise. The cost $c_* \equiv \Delta_*$ seller is indifferent between the two mechanisms, and we call c_* the equilibrium cutoff cost such that measure $p_* = G(c_*)$ sellers with cost lower than c_* runs $\mathrm{A}(\rho_*)$ and measure $1 - p_*$ of the sellers with cost higher than c_* chooses $\mathrm{P}(\phi_*)$.

The total discounted payoff is the expected utilities from participation in auctions and posted prices,

$$\underline{u}_*(v) = \mathbf{1}_{v \geq \rho_*} G(c_*) u(v|\mathrm{A}(\rho_*)) + \mathbf{1}_{v \geq \phi_*} (1 - G(c_*)) u(v|\mathrm{P}(\phi_*))$$

where the payoffs in the auction and the posted price are respectively,

$$\begin{aligned}
u_*(v|\mathrm{A}(\rho_*)) &= \mathbf{1}_{v \geq \rho_*} \left[\int_{\rho_*}^{v} H_*^{n-1}(z) \, dw(z) + \delta s \underline{u}(v) \right] \\
u_*(v|\mathrm{P}(\phi_*)) &= \mathbf{1}_{v \geq \phi_*} \left[\frac{1}{n} \frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)} (v - w(\phi_*)) + \left(1 - \frac{1}{n} \frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)} \right) \delta s \underline{u}(v) \right] \\
&= \mathbf{1}_{v \geq \phi_*} \left[\frac{1}{n} \frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)} (w(v) - w(\rho)) + \delta s \underline{u}(v) \right]
\end{aligned}$$

For value $v \in (\rho_*, \phi_*)$, the derivation is similar to the previous section and only differs by the extra $G(c_*)$ term.

$$u_*(v|\mathrm{A}(\rho_*)) = \frac{1}{1 - G(c_*)\delta s}\int_{\rho_*}^{v} H_*^{n-1}(z)\, dw(z)$$

and

$$w_*'(v) = \frac{1 - s\delta G(c_*)}{1 - s\delta G(c_*) + s\delta G(c_*) H_*^{n-1}(v)}.$$

For $v \geq \phi_*$, the calculation is more convoluted,

$$(1-\delta s)\underline{u}(v) = \int_{\rho_*}^{v} H_*^{n-1}(z)\, dw_*(z) + \frac{1}{n}\frac{1 - H_*^n(\phi_*)}{1 - H_*(\phi_*)}(w_*(v) - w_*(\phi_*)).$$

In summary, an equilibrium $(M_*(\cdot), \sigma_*(\cdot,\cdot), H_*(\cdot), \mu_*(\cdot))$ is characterized by c_*, the cutoff cost, ϕ_*, the optimal posted type, ρ_*, the optimal reserve type, and the stationary value distribution H_*, where

1. Mass p_* of sellers with cost $c \leq c_*$ run the same optimal reserve price auction $\mathrm{A}(\rho_*)$ and the other mass $1-p_*$ of sellers with cost $c > c_*$ uses the same optimal posted price mechanism $\mathrm{P}(\phi_*)$, where ρ_* and ϕ_* are determined by $\mathrm{MR}_*^{\mathrm{A}}(\rho_*) = \mathrm{MR}_*^{\mathrm{P}}(\phi_*) = 0$.

2. Value v buyer's equilibrium WTP is

$$w_*(v) = v - \frac{\delta s\left[\mathbf{1}_{v\geq\rho_*} p_* u(v|\mathrm{A}(\rho_*)) + \mathbf{1}_{v\geq\phi_*}(1-p_*)\, u(v|\mathrm{P}(\phi_*))\right]}{1 - \delta s\left[1 - \mathbf{1}_{v\geq\rho_*} p_* H_*^{n-1}(v) - \mathbf{1}_{v\geq\phi_*}(1-p_*)\frac{1}{n}\frac{1-H_*^n(\phi_*)}{1-H_*(\phi_*)}\right]}$$

3. Stationary value distributions h_* and H_* are characterized by (4.2) and (4.3) with the expected losing probability function $l_*(\cdot)$,

$$l_*(v) = \begin{cases} 1 & v \leq \rho_* \\ p_*(1 - H_*^{n-1}(v)) & v \in (\rho_*, \phi_*] \\ p_*(1 - H_*^{n-1}(v)) + (1-p_*)\left(1 - \frac{1}{n}\frac{1-H_*^n(\phi_*)}{1-H_*(\phi_*)}\right) & v \in (\phi_*, 1] \end{cases}$$

4. The equilibrium belief μ_* is

$$\mu_*\left(H_*\left(\cdot\right),\left(p\circ \mathrm{A}\left(\rho_*\right),\left(1-p\right)\circ \mathrm{P}\left(\phi_*\right)\right),\sigma_*\left(\cdot,\cdot\right)\right)=1.$$

Proof of Proposition 24. Similar to the previous model, the equilibrium is

$$\begin{aligned}
H_*\left(\rho_*\right) &= \left\{1+\frac{s}{1-s}\frac{1}{n}\left[\left(1-H_*^n\left(\rho_*\right)\right)G\left(c_*\right)+\left(1-H_*^n\left(\phi_*\right)\right)\left(1-G\left(c_*\right)\right)\right]\right\}F\left(\rho_*\right)\\
&= \left[1+\frac{s}{1-s}\frac{1}{n}\left(1-H_*^n\left(\rho_*\right)\right)+\frac{s}{1-s}\frac{1}{n}\left(H_*^n\left(\rho_*\right)-H_*^n\left(\phi_*\right)\right)\left(1-G\left(c_*\right)\right)\right]F\left(\rho_*\right)\\
&\equiv \left[1+\frac{s}{1-s}\frac{1}{n}\left(1-H_*^n\left(\rho_*\right)\right)-\epsilon\left(\rho_*\right)\right]F\left(\rho_*\right)
\end{aligned}$$

where $\epsilon\left(\rho_*\right)$ is strictly positive. As before, substitute in the equilibrium condition

$$H_*\left(\rho_*\right)=F\left(\rho_*\right)/\left(F\left(\rho_*\right)+\rho_*f\left(\rho_*\right)\right),$$

$$\frac{1}{F\left(\rho_*\right)+\rho_*f\left(\rho_*\right)}=1+\frac{s}{1-s}\frac{1}{n}\left(1-\left(\frac{F\left(\rho_*\right)}{F\left(\rho_*\right)+\rho_*f\left(\rho_*\right)}\right)^n\right)-\epsilon\left(\rho_*\right). \tag{4.29}$$

Compared with (4.25), the first two terms of RHS are increasing, but with the last term, the curve shifts down and intersects the LHS at a bigger ρ_* than before. However, bigger ρ_* means smaller $H_*\left(\rho_*\right)$ and bigger $1-H_*^n\left(\rho_*\right)$. Therefore, sale probability and efficiency increases for the auctions, but it increases when there are sellers switching to posted prices which have sale efficiency $1-H_*^n\left(\phi_*\right)$, so the total effect is ambiguous. However, more posted price mechanisms bring more allocative inefficiency. □

Chapter 5 An Evolutionary Justification for Overconfidence

Note: This paper has been published in *Economics Letters* in November 2013 (Zhang, 2013a).

Abstract: This paper suggests that the evolutionarily optimal belief of an agent's intrinsic reproductive ability is systematically different from the posterior belief obtained by the perfect Bayesian updating. In particular, the optimal belief depends on how risk-averse the agent is. Although the perfect Bayesian updating remains evolutionarily optimal for a risk-neutral agent, it is not for any other. Specifically, the belief is always positively biased for a risk-averse agent, and the more risk-averse an agent is, the more positively biased the optimally updated belief is. Such biased beliefs align with experimental findings and also offer an alternative explanation to the empirical puzzle that people across the population appear overconfident by consistently overestimating their personal hereditary traits.

Keywords: Non-Bayesian belief, risk-aversion, evolutionary economics, overconfidence, bias

JEL: C73, D83

5.1 Introduction

Daily observations, empirical evidence, and experimental findings suggest that under many circumstances human beings are not perfect Bayesian updaters, especially when they learn about their personal traits such as beauty and intelligence. One particular systematic deviation from the perfect Bayesian updating is **positive bias**: people tend to over-react to positive signals indicating their possibility of high ability and under-react to negative ones indicating otherwise. As a consequence, people are unrealistically overconfident about their personal characteristics.

The fact that the asymmetric updating results in population-wide overestimation of personal ability and persistently hyped beliefs even with overwhelming contrary indications is reflected in studies self-reporting many personal traits. 88% of American drivers believe that they drive more safely than the median driver does (Svenson, 1981) and 75% of Harvard undergraduate students think they have above median IQ among their peers, even with repeated informative signals indicating their true tested results (Möbius et al., 2012). Although many attribute the result to cognitive limitations like selective recall and selective information acquisition, while maintaining the perfect Bayes Rule evidence is too overwhelming and magnitude is too colossal to refuse entertaining the possibility of an updating system other than the perfect Bayesian and a resulting non-Bayesian posterior. This is especially true for beliefs about own traits - it is after all the same group of Harvard undergraduates who show perfect Bayesian updating behavior about other people's abilities in the same study. An increasing number of papers assume people have direct belief utilities because they care about self-esteem (Benabou and Tirole, 2002; Kőszegi, 2006; Eil and Rao, 2011), but this assumption almost directly implies positively biased beliefs.

This paper attempts to justify the seemingly imperfect human behavior - positively biased updates and beliefs - from an evolutionary standpoint, without assuming a belief utility or cognitive deficiencies. If such seemingly imperfect updating behavior is so prevalent, maybe it is key to our survival, or those who have adopted such updating behavior are the fittest and have adapted and survived to this date. The personal traits are esteemed because they affect the reproductive efficiencies, so the belief should be formed in a way that achieves maximal reproductive success. Although an agent cares about the survival of her offspring, she wants to maximize her *utility* about the survival of offspring. When the agent is not risk-neutral, utility maximization is not the same goal as the primary evolutionary goal of maximizing the expected survival rates of offspring. When a principal has an unaligned goal with an agent, the literature in contract theory and mechanism design considers possible compensation schemes by a principal to an agent to align the two goals so that the agent takes the effort level the principal desires. In this paper, the possible financial scheme is impossible, but the principal, in this case Nature, can choose to manipulate the agent's belief about her own ability. Nature achieves his goal of maximized growth while the agent still maximizes her utility. The agents who have the evolutionarily optimal beliefs would have the highest overall expected growth and survival rate. We will show concretely that risk-averse agents exhibit positively biased beliefs, as the way that evolution corrects risk deficiency.

Non-Bayesian updating has been widely studied by psychologists, evolutionary biologists, philosophers, and behavioral economists. This paper, in a broad sense, attempts to explain phenomena observed from psychological and behavioral experiments, by abiding by philosophical rules and utilizing techniques developed by evolutionary economists. Although there are many experiments confirming the non-Bayesian behaviors of human in various settings, the theoretical literature explaining such behaviors is relatively scant and they do not provide or suggest an evolutionary link (Epstein, 2006; Epstein et al., 2008, 2010). There are only a few papers providing evolutionary justifications to risk aversion and Bayesian updating. Okasha (2012) shows that Bayesian updating is evolutionarily optimal when the agents are "rational" in a philosophical sense - essentially having von Neumann-Morgenstern utility. Levy (2010) shows that constant relative risk-averse (CRRA) utility function is evolutionarily optimal if the agent's objective is to have descendants forever.

The result that non-Bayesian updaters and believers dominate the competition sharply contrasts with some rational expectations results in the financial market. In the financial market, those investors who make inaccurate predictions about the market partly due to imperfect Bayesian updating are driven out in the competitive equilibrium (Sandroni, 2000). However, in the reproduction market, the ones who make reproductive decisions from perfectly updated beliefs are the ones who are driven out and will be extinct in the long run.

Section 5.2 defines and characterizes the evolutionarily optimal posterior. Section 5.3 shows that belief is consistently higher than true population proportion of high abilities. Section 5.4 concludes by pointing out limitations of the current model and possible future directions.

5.2 Evolutionarily Optimal Posterior

Let us introduce the setup and demonstrate the key results in a simple model with the imperfectly observable reproductive trait taking two possible values. In particular, we show that the evolutionarily optimal posterior is always higher than the perfect Bayesian posterior, for all signals.

An agent chooses a reproductive action based on her belief about her reproductive trait to maximize her expected utility about the survival of the offspring. The agent's action a and trait x determine the survival rate or the number of her offspring, which we call the reproduction function. The trait can be either high (H) or low (L) but the agent does not directly observe it. Assume the reproduction function $F(a, x)$ is continuously differentiable, increasing, and concave in a, with the boundary condition $F(0, x) = 0$. Furthermore, assume that $F(a, H) > F(a, L)$, i.e., exerting the same effort, an agent with high ability produces more than an agent with low ability. It is increasingly costly to exert effort in mating activities, so the cost function $c(\cdot)$ is assumed to be increasing and weakly convex, $c'(a) > 0$, $c''(a) \geq 0$. The agent derives utility $u(\cdot)$ from the net benefit $F(a, x) - c(a)$ she gets, with $u'(\cdot) > 0$.

Since the agent does not perfectly observe the trait x she has, she forms a posterior belief μ from a signal s coupled with a prior μ_0 inherited possibly from her parent. The signal can be H or L, and the signal generating process is publicly known,

$$\Pr(s = H|x = H) = p_1,$$
$$\Pr(s = H|x = L) = p_2.$$

In particular, the perfect Bayesian posterior μ^B can be expressed as

$$\text{logit}\left(\mu^B\right) = \text{logit}(\mu_0) + \mathbf{1}_{s=H}\lambda_H + \mathbf{1}_{s=L}\lambda_L,$$

where μ^B, μ_0 are shorthands for $\mu^B(H), \mu_0(H)$ as a slight abuse of notation, and $\text{logit}(\mu) = \log\left(\frac{\mu}{1-\mu}\right)$. $\lambda_s = \log\left(\frac{\Pr(s|x=H)}{\Pr(s|x=L)}\right)$ represents the log-likelihood of being high type given signal s. In particular, $\lambda_H = \log\left(\frac{p_1}{p_2}\right)$ and $\lambda_L = \log\left(\frac{1-p_1}{1-p_2}\right)$.

Agent's Problem

Given the posterior belief μ, she chooses the effort level a to maximize expected utility,

$$u_A(a|\mu) = \mu u(F(a, H) - c(a)) + (1-\mu) u(F(a, L) - c(a))$$

So the FOC is

$$0 = \mu u'(F(a, H) - c(a))(F_a(a, H) - c'(a)) + (1-\mu) u'(F(a, L) - c(a))(F_a(a, L) - c'(a))$$

Or rearrange, a is chosen so that the following equation is satisfied,

$$\frac{\mu}{1-\mu} = \left|\frac{F_a(a, L) - c'(a)}{F_a(a, H) - c'(a)}\right| \cdot \frac{u'(F(a, L) - c(a))}{u'(F(a, H) - c(a))} \tag{5.1}$$

Nature's Problem

Consider the agent's problem regarding a reproductive decision. Consider that x is an imperfectly observable reproductive trait (IQ, EQ, psychological fitness, for example), and a is the agent's effort spent in searching and mating with the convex cost of the effort that represents a reduction in one's own survival and fitness (frequenting dating sites such as bars and websites takes nontrivial effort and opportunity cost), with $F(a, x)$ guiding the number (and quality) of offspring an agent produces. The objective of Nature, a perfect Bayesian updater, is then to maximize the overall expected growth of the population,

$$u_N(a) = \mu^B (F(a, H) - c(a)) + \left(1 - \mu^B\right)(F(a, L) - c(a))$$

Its FOC is

$$\frac{\mu^B}{1 - \mu^B} = \left| \frac{F_a(a^*, L) - c'(a^*)}{F_a(a^*, H) - c'(a^*)} \right| \tag{5.2}$$

where a^* is the evolutionarily optimal action Nature wants the agent to take, given prior μ_0 and signal s.

If Nature can manipulate the agent's belief to induce her to choose the evolutionarily optimal action, then (5.1) becomes

$$\frac{\mu^*}{1 - \mu^*} = \left| \frac{F_a(a^*, L) - c'(a^*)}{F_a(a^*, H) - c'(a^*)} \right| \cdot \frac{u'(F(a^*, L) - c(a^*))}{u'(F(a^*, H) - c(a^*))}.$$

Plugging in (5.2),

$$\text{logit}(\mu^*) = \text{logit}\left(\mu^B\right) + \log\left[\frac{u'(F(a^*, L) - c(a^*))}{u'(F(a^*, H) - c(a^*))}\right]. \tag{5.3}$$

Because $F(\cdot, L) < F(\cdot, H)$, when $u(\cdot)$ is concave, the second term on RHS of (5.3) is positive, which we refer to as the **risk-averse bias** $B(a^*)$. The evolutionarily optimal posterior belief of a risk neutral agent is the perfect Bayesian posterior, there is no bias; so it is a bias stemmed from risk aversion of the agent. When the utility function is CRRA or CARA, the bias perfectly correlates with the risk aversion factor, since a^* does not depend on the risk aversion factor but only on Bayesian posterior.

Example 3. If the utility is CRRA, $u(C) = C^{1-\rho}/(1-\rho)$, $\rho \geq 1$, $u'(C) = C^{-\rho}$,

$$B(a^*) = \rho \log \left| \frac{F(a^*, H) - c(a^*)}{F(a^*, L) - c(a^*)} \right|.$$

Example 4. If the utility is CARA, $u(C) = K - \exp(-\alpha C)$, $\alpha \geq 0$, $u'(C) = \alpha \exp(-\alpha C)$,

$$B(a^*) = \alpha [F(a^*, H) - F(a^*, L)].$$

The bias is positive for any risk-averse agent and negative for any risk-loving agent.

Proposition 27. *When the agent is risk-averse (risk-loving), the evolutionarily optimal posterior is a non-Bayesian posterior, positively (negatively) biased towards the high type compared to the perfect Bayesian posterior.*

Proof of Proposition 27. Because $F(\cdot, L) < F(\cdot, H)$ and $u(\cdot)$ is concave (convex),

$$u'(F(a^*, L) - c(a^*)) > (<) u'(F(a^*, H) - c(a^*)).$$

By (5.3),

$$B(a^*) = \log \left[\frac{u'(F(a^*, L) - c(a^*))}{u'(F(a^*, H) - c(a^*))} \right] > (<) \log 1 = 0.$$

□

A few comments are in order. The evolutionarily optimal posterior is invariant to the order of arrival of a stream of i.i.d. signals. The perfect Bayesian posterior is invariant to the order of arrival of a stream of i.i.d. signals, given $\{s_t\}$ and prior μ_0, the posterior μ^B is

$$\text{logit}\left(\mu^B\right) = \text{logit}(\mu_0) + \#\{t : s_t = H\} \lambda_H + \#\{t : s_t = L\} \lambda_L,$$

Since a^* only depends on μ^B, it is invariant to the order of arrival, then by (5.3), μ^* is invariant to the order of arrival of $\{s_t\}$. Therefore, whether the agent makes the reproductive decision after appearance of one or more signals does not affect the reproductive outcome or the personal posterior belief.

As a result of the evolutionary correction, the long run survival of an agent does not depend on her risk aversion factor but only on her true trait. A non-Bayesian belief corrects the possible evolutionary sub-optimal reproductive decision a risk-averse agent can make, and the level of correction depends on the degree of risk aversion so that agents with the same Bayesian posterior make the same reproductive decision. Without sexual production or mutation, only the agents of high type will survive, and the evolutionarily optimal updating makes an agent realize that she is of high type faster, and of low type slower.

5.3 Population Posterior

Investigation of the population evolutionarily optimal posterior belief shows that regardless of the population composition, as long as they are risk-averse, more people believe that they are of high type than there really are. Since we are survivors and winners of millions of years of evolutionary struggles, this result possibly explains the aforementioned findings that people are overconfident about their intrinsic skills.

The result can be directly understood from the asymmetric belief. Every agent believes she is more likely to be a high type than a perfect Bayesian would believe. Regardless of the evidence (stream of signals) a person receives, her belief about herself being a high type is always higher than the Bayesian posterior belief.

Suppose the population is composed of proportion q realized high type and proportion $1-q$ realized low type agents after the previous action. Suppose that after each time an agent takes an action, she observes the outcome of her action and infers her true type (since $F(a,x)$ is bijective, knowing what effort a she exerted and observing F is enough to uncover her true trait x). After each action, there is a probability ϵ that she mutates: her true type switches from one to another. Therefore, an agent's prior μ_0 after an action is $1-\epsilon$ if she is high or ϵ if low.

We construct a population posterior and use it as a criterion to evaluate the percentage of people believing they are of high type. For any posterior $\mu = \{\mu_A\}_A$, the expected population posterior is defined as the total population belief that they are a high type.

$$q(\mu) = \int_A \mu_A dA$$

If every agent is a perfect Bayesian, given the signal generating process, their population posterior should be the same as the population prior, which is the same as the population composition, q, $q^B \equiv q\left(\mu^B\right) = q$. On the other hand, any agent risk-averse A has $\mu_A^* > \mu_A^B$, so in population, $q(\mu^*) > q\left(\mu^B\right)$, a relatively small portion of risk-seeking agents will not alter the population belief.

Proposition 28. *If most agents are risk-averse in the population, the evolutionarily optimal population posterior belief about high type is strictly greater than the population composition of high type.*

Even though every agent knows that in the population, there is only a proportion q of high type agents, the aggregate of individual beliefs is higher than it. This finding explains the perverse scenario mentioned in Introduction that objective aggregates of desirable personal characteristics are always lower than their subjective individual reports. The key to the result is the imperfect observability of personal characteristics and possibility of mutation.

5.4 Conclusion

The paper shows that in order to maximize the expected number of offspring, an agent with nonlinear preference has a belief different from the belief obtained by perfect Bayesian updating. In particular, for any risk-averse agent, she thinks more highly of herself than she does if she is a perfect Bayesian. Therefore, the results suggest that evolution and survival play a role in the widespread existence of non-Bayesian belief, especially about a person's own trait that influences reproductive decisions.

Although the paper provides a possible evolutionary channel to persistent overconfidence across the population, it fails to characterize the exact way how this optimal overconfidence is sustained. In terms of terminology of the model, the paper is able to rationalize the misalignment of the optimal posterior and the perfect Bayesian posterior, but it fails to characterize the updating *rule*[1] that can consistently achieve and sustain the optimal posterior. The problem is especially conspicuous when the agent updates after each of many sequentially observed noisy signals. Characterizing or approximating such evolutionarily optimal updating rule would be interesting and useful.

It is also interesting to explore why and how risk aversion and non-Bayesian belief/updating rule could be evolutionarily optimal at the same time. If we treat the objective of having descendants forever to be the goal for each individual as in Levy (2010) and the objective of maximizing expected population growth to be the goal for the entire group, an evolutionarily optimal non-Bayesian updating is justified. While each individual agent needs to be risk-averse to have descendants forever, the ones who dominate the population are those following a non-Bayesian updating rule that, coupled with risk-averse utility, maximizes the expected number of descendants in each generation.

Finally, a model with multiple signals and/or attributes may be more realistic and may help generate more insights including but not limited to conservative updating.

[1] I thank a referee for emphasizing the difference between an updating rule and an updated posterior.

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