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The optimal sequence of prices and auctions

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ABSTRACT

To sell a good before a deadline, a monopolist chooses between posting a price and running a costly reserve-price auction each period. Buyers with independent private values arrive over time. For a wide range of auction costs, the profit-maximizing mechanism sequence is to post prices first and then to run auctions. The optimality of the prices-then-auctions mechanism sequence provides a new justification for the hybrid sales mechanism of allowing the “Buy It Now” option before a standard auction on eBay.

1. Introduction

In theory, an auction with a carefully chosen reserve price maximizes the expected revenue when buyers have independent private values (Myerson, 1981). In practice, running an auction can be more costly than posting a price.

For example, on eBay, auction-style listings cost more than price-style listings. Available pricing choices on eBay are auction, Buy It Now (posted price), and hybrid Buy It Now auction (a “Buy It Now” price option before the auction starts). Additional costs associated with an auction include *reserve price fee, listing upgrade fees*, and *duplicate listing fee*.[2] The *reserve price fee* is applied if a positive reserve price is set. This fee is the higher of \$5 and 7.5% of the reserve price, capped at \$250; the minimum, percentage, and maximum have increased over time. The *listing upgrade fees* include (i) \$1 to have the auction last one or three days—rather than the default seven days—a preferred auction length because buyers more frequently search auctions that are closer to the deadline, and (ii) \$0.70 (\$0.35 for a more expensive listing) to list in the Collectibles, Art, Pottery & Glass, and Antiques categories—some of the most popular categories for auctions.[3] The *duplicate*

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[1] This paper is a revised version of Chapter 3 of my Ph.D. dissertation at the University of Chicago (Zhang, 2015a). The extended abstract of the paper appears in the Proceedings of the Third Conference on Auctions, Market Mechanisms, and Their Applications (Zhang, 2015b). I thank Scott Kominers and Phil Reny for continuous advice, Alex Frankel, Johannes Hörner, Fei Li, and Richard van Weelden, for helpful suggestions. I have benefited from the discussions at Stony Brook Game Theory Festival, Midwest Theory Conference, Econometric Society World Congress, and University of Chicago seminars. The financial support by the Yahoo! Key Scientific Challenges Fellowship, Graduate Student Affairs Travel Fund, and the National Science Foundation, and the hospitality of Microsoft Research New England are gratefully acknowledged. I thank Jacob Schmitter for excellent research assistance.

[2] Appendix A shows the screenshots of options when listing an auction, a posted price, and a hybrid Buy It Now auction on eBay, respectively.

[3] For additional details, see selling fees on eBay, https://www.ebay.com/help/selling/fees-credits-invoices/selling-fees. Other fees that auctions and “Good Till Canceled” price listings both face—but fixed-length price listings do not face—include (i) \$2–\$6 to bold the title, (ii) \$1.5–\$6 to add a subtitle, (iii) \$0.1-\$0.6 to list designer, (iv) \$0.1-\$0.5 for international site visibility, and (v) fees to list in two categories.

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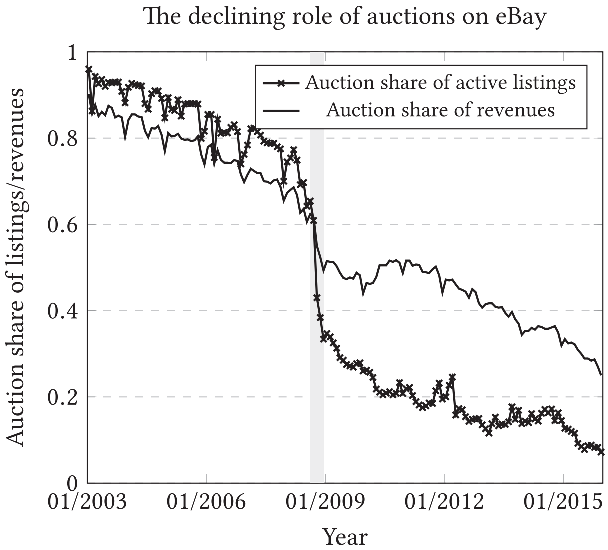

Fig. 1. The declining role of auctions on eBay from 2003 to 2015; reproduced from Einav et al. (2018). The figure shows each month's average daily share of active eBay listings and revenues from January 2003 to December 2015, comparing pure auctions listings to all pure auction and posted prices listings; hybrid Buy It Now auctions are excluded. During that time period, the auction share of active listings has decreased from 96% to 7.2%, and the auction share of revenues has decreased from 90.2% to 25.0%. After posted prices were allowed to be permanent—i.e., "Good Till Canceled" listings—in September 2008, the listing and revenue shares of auctions drastically declined (from 65.4% in August to 33.4% in December and from 62.4% to 49.3%, respectively).

listing fee associated with listing duplicate identical auction-style listings varies by the item but is used to allow the buyer to have different buying options, whether it be different types of auctions or fixed price listings. Altogether, these fees accumulate to a considerably higher operational cost for auctions than for posted prices.[4]

Evidence suggests that these additional costs associated with auctions fundamentally change eBay sellers' pricing choices. Fig. 1 illustrates the gradually declining role of auctions, relative to prices, on eBay from January 2003 to December 2015. During that time span, the auction share of active listings decreased from 96% to 7.2%, and the auction share of revenue decreased from 90.2% to 25.0%. When posted prices were allowed to be permanent—i.e., "Good Till Canceled" listings—in September 2008, the listing and revenue shares of auctions drastically declined (from 65.4% in August to 33.4% in December and from 62.4% to 49.3%, respectively). Einav et al. (2018) use granular data from 2003 to 2009 to show that the declined role of auctions can be attributed to the change in sellers' incentives rather than the change in seller composition. The key trade-off, as they argue, is that "auctions enable price discovery and buyer competition, but are less convenient for buyers."

This paper takes into account auction costs for a seller and investigates how she repeatedly chooses between prices and auctions to maximize her expected profit. Specifically, a seller sells one unit of an indivisible good within T periods, and in each period, she either runs a reserve-price auction incurring a per-period auction cost or posts a price for free. The deadline T can be finite, infinite, or stochastic. Buyers with independent private values enter the market each period and pay a cost to bid in an auction. Buyers can be short-lived or long-lived, myopic or forward-looking.

Most interestingly, when the good has to be sold before a deadline and when the auction cost is within an intermediate range of auction cost, the optimal mechanism sequence takes a neat form: prices then auctions. No other combination of prices and auctions, though feasible, is optimal; it is never optimal to run auctions then to post prices, or to alternate between prices and auctions. The prices-then-auctions mechanism sequence resembles the hybrid Buy It Now auction: the seller posts a price at which any buyer can snatch the good before an auction starts. Empirically, such a selling format accounted for a significant amount of listings on eBay: Of the 7382 baseball tickets listed for 2007 regular season Cincinnati Reds home games, 38% (2794) were sold by the Buy It Now auctions while 50% (3698) were sold by auctions and 12% (890) by prices (Bauner, 2015).

The main result, the optimality of the prices-then-auctions sequence, relies on the endogenously higher opportunity cost of an auction in the dynamic setting. In the static setting, the seller only faces the trade-off between (i) a higher revenue for the optimal auction than for the optimal posted price and (ii) a higher cost for an auction than for a posted price. In the dynamic setting, the seller faces an additional cost when she uses an auction. The good not sold today is worth the

[4] Besides the monetary cost imposed by the platform, an auction is also more complex than a posted price (Hart and Nisan, 2013). Despite its relevance, the intricacies of this complexity cost—an operational cost for sellers and a mental cost for buyers—are outside the scope of this paper. I focus on the consequences of additional monetary cost of the auction. In addition, another advantage of posted prices compared to auctions is their immediacy; this immediacy can be thought to be parsimoniously captured by the additional cost associated with auction (Mathews, 2004). Posted prices are preferred to auctions for other reasons, too: Federal Reserve uses posted price (i.e., Discount Window) to lend to banks, and only uses an auction during a crisis (Hu and Zhang, 2020).

expected revenue it generates in the next period. The probability of sale is higher using the optimal auction than using the optimal posted price, because the optimal posted price is always higher than the optimal reserve price.[5] Consequently, using an auction in a dynamic setting incurs not only an operational cost but also a higher opportunity cost associated with selling the good early. The operational cost stays constant but the opportunity cost decreases over time. Therefore, running an auction in a later period incurs a lower combined operational and opportunity cost than doing so in an earlier period. If an auction is ever used, it is used in later periods.

Having understood the auction's endogenous declining opportunity cost, we can easily see that the optimal prices-then-auctions sequence persists in more general settings. The optimality of the prices-then-auctions sequence holds even when the sale deadline is stochastic, the seller becomes increasingly impatient, the seller faces declining auction costs, buyers arrive stochastically and have sequential outside options, the seller faces separate markets for prices and auctions, the mechanism designer is procuring a contract rather than selling, and the seller sequentially sells multiple goods.

Furthermore, the price-then-auction sequence is also shown to be optimal in two-period settings in which the buyers are long-lived and forward-looking. The key complication with long-lived buyers is that the seller's dynamic problem does not seem to be able to be dissected into per-period problems because each period's mechanism affects buyers' strategies in current and subsequent periods. Using the results in Crémer et al. (2007) and Lee and Li (2020), we can think of the monopolist's dynamic sales problem with short-lived buyers as an optimal search problem à la Weitzman (1979), but this approach also seems to fall short of characterizing the problem with long-lived buyers.

When the horizon is infinite so that there is no exogenous deadline to sell the good, the problem the seller faces in each period is stationary, so the seller runs the same mechanism repeatedly in each period. A high-cost seller posts a constant price and a low-cost seller runs auctions with a constant reserve price. An interesting comparative statics result is derived: More patient sellers are more likely to post prices. This comparative statics sheds light on the issue about the declining use of auctions on eBay (Bajari and Hortacsu, 2004; Einav et al., 2018). It shows that changes in market size and market transaction alone, without changes in seller or buyer preferences, can shift the sellers' choice of mechanisms.

The setup intends to capture an individual seller's problem in a large market such as eBay. For example, a person who has bought a lyric opera ticket but could no longer attend the event scheduled in two weeks chooses between posting a fixed price for the ticket and auctioning off the ticket before it loses its value. The item's posting can last for a week (e.g., it stays as a new item for a week on the front page of the website, the much more likely place for buyers to search). Potential buyers browse the website and encounter the posting for the item on sale, and decide whether or not to buy the ticket immediately. They have idiosyncratic values for the ticket. Although their individual values are unknown, their aggregate demand curve is known to the seller. Buyers are anonymous to the seller so the seller is restricted to use either prices or auctions (Zhang, Forthcoming).

The paper contributes to the theoretical literature that treats the participation cost as a sunk cost borne by the seller. The literature formulates the mechanism design problem as an optimal search problem given asymmetric information, with the seller's cost playing the role of the search cost: McAfee and McMillan (1988), Burguet (1996), Crémer et al. (2007), Crémer et al. (2009) and Lee and Li (2020). This paper departs from the literature by assuming that auctions are costly to the seller while posted prices are not; compared to Crémer et al. (2007) that treats all mechanisms to be costless, this paper considers the setting in which only posted prices are costless. Most importantly, most of the literature—with the exception of Lee and Li (2020)—consider only long-lived buyers, while the main model in this paper considers short-lived buyers.

The optimal price-then-auction mechanism sequence resembles the Buy It Now auction, and provides a justification for its use. The declining opportunity cost that generates such a sequence has not been suggested as an explanation for the use of Buy It Now auction. Previous explanations for the use of Buy It Now auction include reasons for both sellers and buyers. Reasons for sellers include increased revenue due to risk aversive buyers (Budish and Takeyama, 2001), their impatience and risk aversion (Hammond, 2013), increased competition forcing prices down using the Buy It Now feature (Anwar and Zheng, 2015). Reasons for buyers include impatience in waiting for an auction to end (Mathews, 2004), experienced buyers' ability to recognize Buy It Now prices below market price (Standifird et al., 2005), risk aversion of the buyer (Reynolds and Wooders, 2009), and heterogeneous preferences in listing styles (Bauner, 2015).

The rest of the paper is organized as follows. Section 2 introduces the basic setup. Section 3 solves the seller's static problem. Section 4 demonstrates the key insight of the paper with a two-period example. Section 5 solves the seller's problem in the finite horizon when buyers are short-lived. Section 6 discusses the finite-horizon problem when buyers are long-lived. Section 7 concludes. Appendix A presents screenshots of options when an item is listed for sale using different mechanisms on eBay. Appendix B demonstrates the robustness of the result when buyers are short-lived. Appendix C solves and discusses the seller's problem without a deadline. Appendix D provides the proofs in the setting with long-lived buyers.

2. Basic setup

A (female) seller wants to sell one unit of an indivisible good for which she has zero consumption value. She must sell the good within T periods, where T can be one, finite, stochastic, or infinite. She discounts each period by the same

[5] The higher sale probability by auctions is in line with empirical findings. Of the baseball tickets investigated by Bauner (2015), the sale probability is 0.589 by auctions, 0.215 by posted prices, and 0.361 by Buy It Now auctions.

discount factor $\delta \in [0, 1]$. In each period t, n one-period-lived (male) buyers enter the market. Each buyer has a private value v independently drawn from the identical value distribution F with positive density f on the entire support $[0, 1]$. All the agents are risk-neutral and have quasi-linear preferences in transfers. Everything is common knowledge except for the private value each buyer is born with.

Throughout the paper we maintain the assumption that the virtual utility function is increasing. The sole purpose of the assumption is to guarantee that the optimal reserve price and the optimal posted price are uniquely determined so that we do not have to deal with the complication that the optimal mechanism involves ironing. I will use auction's virtual utility (Myerson, 1981) and marginal revenue curve (Bulow and Roberts, 1989) interchangeably to refer to $\alpha(v)$.

Assumption 1. The virtual utility $\alpha(v) \equiv v - (1 - F(v))/f(v)$ is increasing in v.

At the beginning of each period t, the seller chooses a mechanism m_t, either a posted price or any mechanism that is not a posted price. It is free to post a price, and it costs $c \geq 0$ to run any mechanism that is not a posted price. By Myerson (1981), when all mechanisms cost the same, it is optimal to run a standard (second-price) auction with a reserve price. Hence, without the loss of generality, we can focus on the comparison between the posted price and auction. In a posted price mechanism P_p, the seller posts a fixed price p at the beginning of a period and the buyers with values higher than p have equal chances of receiving the good.

Let $(m_\tau, \cdots, m_T)$ denote the *mechanism sequence* of the seller who runs mechanism m_t in period t if the good has not been sold by the end of period $t-1$. The seller's problem is to choose the optimal mechanism sequence $\mathbf{m}^* \equiv (m_1, \cdots, m_T)$ so that the expected profit from running $(m_\tau, \cdots, m_T)$ is maximized for any period τ. Since the buyer arrival process is known and there is no learning by the seller, the seller essentially chooses a sequence of mechanisms at the beginning of the first period, to be executed in each period if the good has not been sold.

3. Static problem

I solve the seller's one-period problem as a building block for the subsequent multi-period problem. Myerson (1981) solved the optimal reserve price auction. I state the characterization of the optimal posted price in a parallel way. The introduction of posted price marginal revenue curve facilitates the exposition and the solution of the seller's problem in the dynamic setting.

The realized revenue of an auction A_r is r if the second highest bid is lower than r, and is v if the second highest bid v is greater than r. Therefore, the expected revenue of an auction is

$$R(A_r) = rn[1 - F(r)]F^{n-1}(r) + \int_r^1 vn(n-1)[1 - F(v)]F^{n-2}(v)f(v)dv,$$

which can be rearranged as

$$R(A_r) = \int_r^1 \alpha(v)dF^n(v).$$

The expected revenue maximizing mechanism among all direct revelation mechanisms is a reserve price auction with the optimal reserve price r^* uniquely determined by $\alpha(r^*) = 0$.[6]

Posting price p results in revenue p if there is a buyer who values it more than p and 0 if no buyer values it more than p. Its expected revenue can be written as $R(P_p) = p[1 - F^n(p)]$. As Bulow and Roberts (1989) construct an auction marginal revenue curve $\alpha(v)$, I construct here a posted price marginal revenue curve. In contrast to the auction marginal revenue curve that is constructed for each buyer with value drawn from distribution $F(v)$, the posted price marginal revenue curve is constructed for the highest value buyer out of the n buyers. Note that the seller generates the same revenue from posting a price to all buyers and from posting the same price to the highest value buyer.[7] The posted price marginal revenue curve is the auction marginal revenue curve with respect to a buyer who draws his value from the first-order distribution F^n,

$$\rho(v) = v - \frac{1 - F^n(v)}{[F^n(v)]'}.$$

We can similarly write posted price P_p's expected revenue as

$$R(P_p) = \int_p^1 \rho(v)dF^n(v).$$

[6] Since $\alpha(v)$ is continuous, $\alpha(0) < 0$ and $\alpha(1) = 1$, r^* exists. Since F satisfies the monotone hazard rate condition, $\alpha(v)$ is strictly increasing, and r^* is uniquely determined. The probability the good is not sold is $k(r^*) = F^n(r^*)$.

[7] At least one buyer is willing to pay price p if and only if the highest value buyer is willing to pay price p. The highest value is drawn from the first-order distribution $F^n(v)$. The highest value buyer's inverse demand curve is $v(q) = (F^n)^{-1}(1 - q)$, and the marginal revenue is derived from $d[q \cdot (F^n)^{-1}(1 - q)]/dq$.

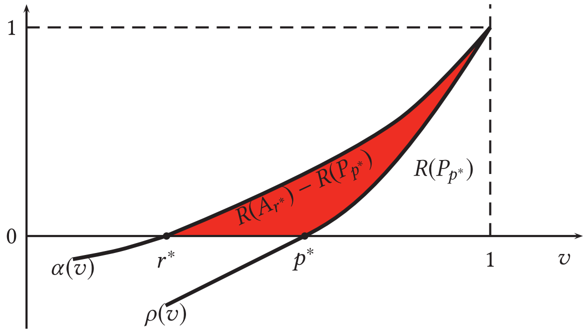

Fig. 2. The marginal revenues and expected revenues in the static problem. An auction marginal revenue curve is $\alpha(v) = v - [1 - F(v)]/f(v)$ and a posted price marginal revenue curve is $\rho(v) = v - [1 - F^n(v)]/[F^n(v)]'$. The optimal reserve price is $r^* = \alpha^{-1}(0)$ and the optimal posted price is $p^* = \rho^{-1}(0)$. The optimal auction's expected revenue is $R(A_{r^*}) = \int_{r^*}^1 \alpha(v) dF^n(v)$ and the optimal posted price's expected revenue is $R(P_{p^*}) = \int_{p^*}^1 \rho(v) dF^n(v)$. The difference between the optimal revenues can be written as $R(A_{r^*}) - R(P_{p^*}) = \int_0^1 xd[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$, and is depicted by the shaeded area in the figure (when the value distribution is uniform).

The optimal posted price p^* is uniquely determined by $\rho(p^*) = 0$.[8]

Such a representation helps us see the similarities between the two classes of mechanisms. More importantly, the representation facilitates the exposition and eases our understanding of the optimal price determination in the dynamic setting. Fig. 2 provides an illustration of the two marginal revenue curves. The optimal reserve price and the optimal posted price equate the marginal revenue to zero. The areas under the curves (weighted with respect to $dF^n(v)$) depict the expected revenues of the two mechanisms.

Since

$$\rho(v) = v - \frac{F^{n-1}(v) + \cdots + 1}{nF^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)} = \alpha(v) + \left[1 - \frac{F^{n-1}(v) + \cdots + 1}{nF^{n-1}(v)}\right] \frac{1 - F(v)}{f(v)},$$

$\rho(v)$ is always smaller than $\alpha(v)$. Since $\alpha(r^*) = \rho(p^*) = 0$, the optimal posted price is always higher than the optimal reserve price. Clearly from Fig. 2, the optimal posted price's expected revenue is smaller. Since $\alpha(r^*) = \rho(p^*) = 0$ and $\alpha(1) = \rho(1) = 1$, by change of variables, the optimal revenue difference can be expressed as

$$R(A_{r^*}) - R(P_{p^*}) = \int_0^1 xd\big[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\big]. \tag{1}$$

Although the optimal reserve price auction generates a higher revenue, the optimal posted price can generate a higher profit if there is a sufficiently high cost of running the auction. In general, running an auction is more appealing for the seller if her cost is lower than a cutoff cost c^* and posting a price is more appealing if her cost is higher than the cutoff cost. The cutoff cost equals the optimal revenue difference expressed in Eq. (1). Proposition 1 characterizes the seller's optimal mechanism in the static setting.

Proposition 1. *Suppose $T = 1$. Let r^* and p^* be the unique solutions to $\alpha(r^*) = \rho(p^*) = 0$ and $c^* = R(A_{r^*}) - R(P_{p^*})$ in Eq. (1). The seller's optimal mechanism is A_{r^*} if $c < c^*$; is P_{p^*} if $c > c^*$; and is A_{r^*} and P_{p^*} if $c = c^*$.*

4. Two-period problem with short-lived buyers

For simplicity, suppose that the seller does not discount ($\delta = 1$). Her objective is to choose a selling mechanism m_1 in the first period and a selling mechanism m_2 in the second period to maximize her expected profit $\pi(m_1, m_2) = \pi(m_1) + k(m_1)\pi(m_2)$, where $k(m_1)$ is the probability that the good is kept to the second period (i.e., not sold after mechanism m_1 in the first period).

The seller's profit-maximizing mechanism sequence (m_1^*, m_2^*) can be solved by backward induction. The seller's problem in the last period is essentially a static problem, and the solution of the static problem, described in the previous section, is found by determining the optimal price and the optimal reserve price, and comparing the revenue from the optimal price and the optimal auction *when the continuation value is zero*. In contrast, the seller's problem in the first period is solving the

[8] The optimal price p^* exists because $\rho(v)$ is continuous, $\rho(0) < 0$, and $\rho(1) = 1$. The optimal price p^* is unique because

$$\rho(v) = v - \frac{1 - F^n(v)}{[F^n(v)]'} = v - \frac{F^{n-1}(v) + \cdots + 1}{nF^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)}$$

is strictly increasing in v, as $[F^{n-1}(v) + \cdots + 1]/[nF^{n-1}(v)]$ is strictly decreasing in v, and $[1 - F(v)]/f(v)$ is decreasing in v by Assumption 1. The probability the good is not sold is $k(p^*) = F^n(p^*)$.

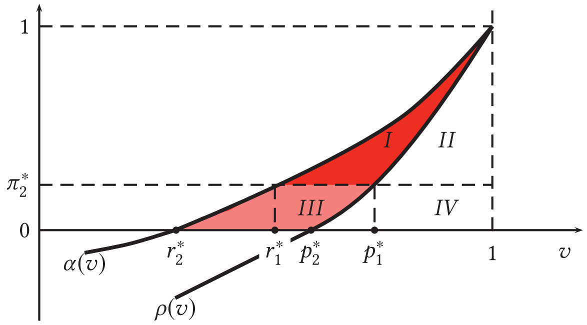

Fig. 3. The marginal revenue curves and the expected revenues in the two-period setting. The revenue difference between the optimal auction and the optimal price is larger in the second period (the darkly sshsh shaded region *I* plus the lightly shaded region *III*) than in the first period (the light red region *III*).

optimal price and the optimal reserve price and comparing the optimal revenues *when the continuation value is* $\pi_2^*(c)$, *the optimal revenue in the second period*. As a result of different effective valuations for the good in the two periods, the seller chooses a higher reserve price and posted price in the first period than in the second period. Consider Fig. 3. The net gain from holding the optimal auction rather than charging the optimal posted price (the dark red region *I*) is smaller than the counterpart in the static problem (the dark red region *I* and light red region *III*).

Because of the declining retention value of the good over time, the revenue advantage of the optimal auction over the optimal posted price increases. To see this, let c_t^* be the cutoff cost in period t such that the seller is indifferent between the optimal auction and the optimal price, and let π_2^* be the optimal profit in the second period. The cutoff cost $c_2^* = \int_0^1 (x-0)d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$, on one hand, is simply the revenue difference between the optimal static auction and the optimal static posted price. The cutoff cost $c_1^* = \int_{\pi_2^*}^1 (x-\pi_2^*)d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$, on the other hand, consists of two terms. The first term, $\int_{\pi_2^*}^1 xd[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))]$, is the difference in the first-period expected revenue between the optimal first-period auction and the optimal first-period posted price. In Fig. 3, the optimal first-period auction's revenue is areas $I+II+III+IV$, and the optimal first-period posted price's revenue is areas $II+IV$, so the revenue difference is areas $I+III$. The second term, $\int_{\pi_2^*}^1 \pi_2^* d[F^n\left(\alpha^{-1}(x)\right) - F^n\left(\rho^{-1}(x)\right)]$, is the difference in the opportunity costs between the optimal first-period auction and the optimal first-period posted price. The opportunity cost of an optimal auction is the probability of no sale in the first period times the expected profit from retaining the good for anther period, which generates the expected profit of $F^n(r_1^*)\pi_2^*$, areas $III+IV$. The opportunity cost of selling using optimal posted price is also the probability of no sale times expected profit generated in the second period, so is $F^n(p_1^*)\pi_2^*$, area IV. Together, the opportunity cost difference is area IV. On net, the total expected revenue difference between the optimal first-period auction and the optimal first-period price is only area *I*, the darkly shaded region.

The optimal prices adjust downward over time. Mathematically, we can easily see from the optimal price determination: the optimal reserve prices are determined by $r_2^* = \alpha^{-1}(0)$ and $r_1^* = \alpha^{-1}(\pi_2^*)$ and the optimal posted prices are determined by $p_2^* = \rho^{-1}(0)$ and $p_1^* = \rho^{-1}(\pi_2^*)$. Economically, the optimal prices equate the marginal revenue to the opportunity cost of selling the good. Since the opportunity cost of selling the good decreases from π_2^* in the first period to 0 in the second period, the optimal reserve and posted prices also decrease accordingly.

The rest of the paper is geared toward showing the robustness of the optimal mechanism sequence in more general settings: for sufficiently low auction costs, a sequence of auctions with declining reserve prices is optimal; for sufficiently high auction costs, a sequence of declining prices is optimal; and for intermediate auction costs, a sequence of declining prices followed by a sequence of auctions with declining reserve prices is optimal. Any mechanism sequence involving auctions followed by prices is never optimal. The optimal sequence of auctions for sufficiently low costs and the optimal sequence of prices for sufficiently high costs are not surprising as they arise almost trivially from the assumption that auction costs more. However, for intermediate auctioning costs, it is not straightforward to arrive at the conclusion that such a nice sequence of mechanisms is the only possible optimal sequence, as any combination of prices and auctions in any order is feasible.

4.1. A two-period example

To illustrate, I present a two-buyer two-period example, which a reader can skip without loss. Suppose that there are two buyers in each of the two periods. Their values are independently drawn from uniform [0,1] distribution, that is, $F(v) = v$ and $f(v) = 1$. The optimal price to post in the second period is $p_2^* = \sqrt{3}/3 \approx 0.577$, determined by $\rho(p_2^*) = p_2^* - \frac{1-(p_2^*)^2}{2p_2^*} = 0$, and the optimal revenue is $R(P_{\sqrt{3}/3}) = \pi(P_{\sqrt{3}/3}) = 2\sqrt{3}/9 \approx 0.385$. The optimal reserve price is $r_2^* = 1/2$, determined by $\alpha(r_2^*) = 2r_2^* - 1 = 0$, and the optimal revenue is $R(A_{r_2^*}) = \int_{r_2^*}^1 (2v-1)dv^2 = 5/12 \approx 0.417$. By Eq. (1), the optimal revenue difference is

$R(A_{r_2^*}) - R(P_{p_2^*})$, which is

$$c_2^* \equiv \int_0^1 x d\left[\left(\alpha^{-1}(x)\right)^2 - \left(\rho^{-1}(x)\right)^2\right] = \frac{5}{12} - \frac{2\sqrt{3}}{9} \approx 0.032. \tag{2}$$

By Proposition 1, the optimal mechanism is $A_{r_2^*} = A_{0.5}$ if $c < c_2^*$, and is $P_{p^*} = P_{\sqrt{3}/3}$ if $c > c_2^*$.

Having solved the second period's optimal mechanism, we can solve for the first period's optimal mechanism. Let $\pi_2^* = \pi(m_2^*) > 0$ denote the optimal second period profit. The total expected profit from posting price p_1 in the first period and using m_2^* in the second period is the expected profit posting price p_1 plus the expected profit using m_2^* in case the good is not sold. Since the probability that the good is not sold in the first period is p_1^2, $\pi(P_{p_1}, m_2^*) = R(P_{p_1}) + p_1^2\pi_2^*$. which can be written as

$$\pi(P_{p_1}, m_2^*) = \int_{p_1}^1 \rho(v)dv^2 + \pi_2^* - \int_{p_1}^1 \pi_2^* dv^2 = \int_{p_1}^1 [\rho(v) - \pi_2^*]dv^2 + \pi_2^*.$$

The optimal price is thus determined by $\rho(p_1^*) = p_1^* - [1 - (p_1^*)^2]/2p_1^* = \pi_2^*$; the optimal posted price is set so that the posted price's marginal revenue equates the opportunity cost of selling the good. Solving for the optimal posted price, we get $p_1^* = (\pi_2^* + \sqrt{(\pi_2^*)^2 + 3})/3$. The total expected profit from running an auction A_{r_1} in the first period is

$$\pi(A_{r_1}, m_2^*) = R(A_{r_1}) - c + r_1^2\pi_2^* = \int_{r_1}^1 [\alpha(v) - \pi_2^*]dv^2 + \pi_2^* - c.$$

The optimal reserve price is determined by $\alpha(r_1^*) = r_1^* - (1 - r_1^*) = \pi_2^*$; the optimal reserve price is set so that the auction's marginal revenue is equated to the opportunity cost of selling the good. Solving for the optimal reserve price, $r_1^* = (\pi_2^* + 1)/2$. Let c_1^* satisfy $\pi(P_{p_1^*}, m_2^*) = \pi(A_{r_1^*}, m_2^*)$; a cost c_1^* seller is indifferent between $P_{p_1^*}$ and $A_{r_1^*}$ in the first period.

$$\begin{aligned} c_1^* &= \left[\int_{r_1}^1 [\alpha(v) - \pi_2^*]dv^2 + \pi_2^*\right] - \left[\int_{p_1}^1 [\rho(v) - \pi_2^*]dv^2 + \pi_2^*\right] \\ &= \int_{r_1^*}^1 [\alpha(v) - \pi_2^*]dv^2 - \int_{p_1^*}^1 [\rho(v) - \pi_2^*]dv^2. \end{aligned}$$

But remember that $\alpha(r_1^*) = \rho(p_1^*) = \pi_2^*$. With the same change of variable performed for Eq. (1),

$$c_1^* = \int_{\pi_2^*}^1 x d\left[\left(\alpha^{-1}(x)\right)^2 - \left(\rho^{-1}(x)\right)^2\right] - \int_{\pi_2^*}^1 \pi_2^* d\left[\left(\alpha^{-1}(x)\right)^2 - \left(\rho^{-1}(x)\right)^2\right]. \tag{3}$$

$c_1^* \approx 0.0075$. When $c < c_1^*$, auction $A_{r_1^*}$ is used, and when $c \geq c_1^*$, posted price $P_{p_1^*}$ is used.

Comparing Eqs. (2) and (3), we can easily see that $c_1^* < c_2^*$. Therefore, the optimal mechanism sequence can be characterized as follows. When $c < c_1^*$, the optimal mechanism sequence is an auction in the first period followed by another in the second period. If $c_1^* \leq c < c_2^*$, the optimal mechanism sequence is a price in the first period followed by an auction in the second period. If $c \geq c_2^*$, the optimal mechanism sequence is a price in the first period followed by a lower price in the second period. The optimal mechanism's optimal reserve price and optimal posted price are as illustrated in Fig. 4. Although auction-auction, auction-price, price-price, and price-auction sequences are all feasible, an auction-price sequence is never optimal.

5. Finite-horizon problem with short-lived buyers

In this section, we solve the seller's profit maximization problem when T is finite and fixed. The seller's discounted sum of payoffs at period $t < T$ is her expected profit in the current period plus her discounted payoff if the good is not sold in the current period,

$$\pi(m_t, m_{t+1}, \cdots, m_T) = \pi(m_t) + k(m_t)\delta\pi(m_{t+1}, \cdots, m_T).$$

Theorem 1. *Suppose T is finite. The optimal mechanism sequence for a cost c seller is characterized by a sequence of strictly increasing cutoff costs, $c_1^* < c_2^* < \cdots < c_T^*$, such that in period t an auction is optimal if $c \leq c_t^*$ and a posted price is optimal if $c \geq c_t^*$. Therefore, the optimal mechanism sequence is a sequence of auctions when the auction cost is smaller than c_1^*, is a sequence of prices then a sequence of auctions when the auction cost is between c_1^* and c_T^*, and is a sequence of prices when the auction cost is bigger than c_T^*.*

By Principle of Optimality, we can solve the problem backwards. We have already solved period T's problem, as it has the same solution as the one-period problem. We restate Proposition 1 in the T-period problem.

Lemma 1. *Suppose T is finite. Let $r_T^* = \alpha^{-1}(0)$, $p_T^* = \rho^{-1}(0)$, and $c_T^* = R(A_{r_T^*}) - R(P_{p_T^*})$. The seller's optimal mechanism in period T is $m_T^*(c) = A_{r_T^*}$ if $c < c_T^*$, $m_T^*(c) = P_{p_T^*}$ if $c > c_T^*$, and $m_T^*(c) = A_{r_T^*} = P_{p_T^*}$ if $c = c_T^*$.*

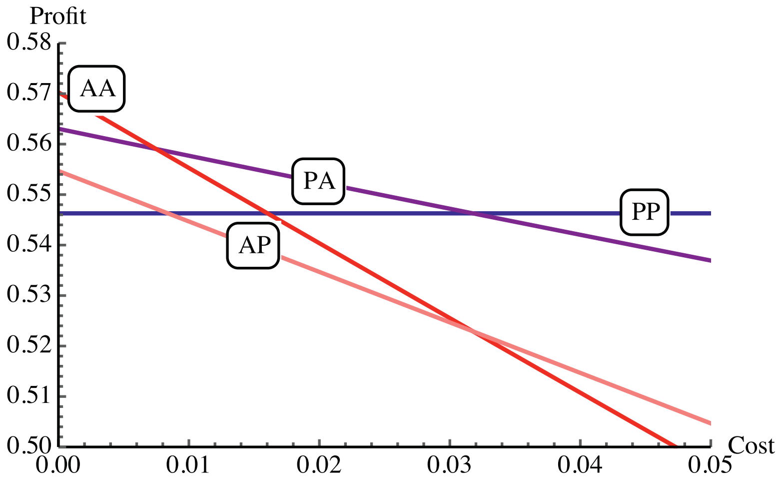

Fig. 4. The optimal mechanism sequence's revenue as the auction cost varies. When the auction cost is smaller than 0.0075, the optimal mechanism sequence is auction-auction. When the auction cost is between 0.0075 and 0.032, the optimal mechanism sequence is price-auction. When the auction cost is bigger than 0.032, the optimal mechanism sequence is price-price.

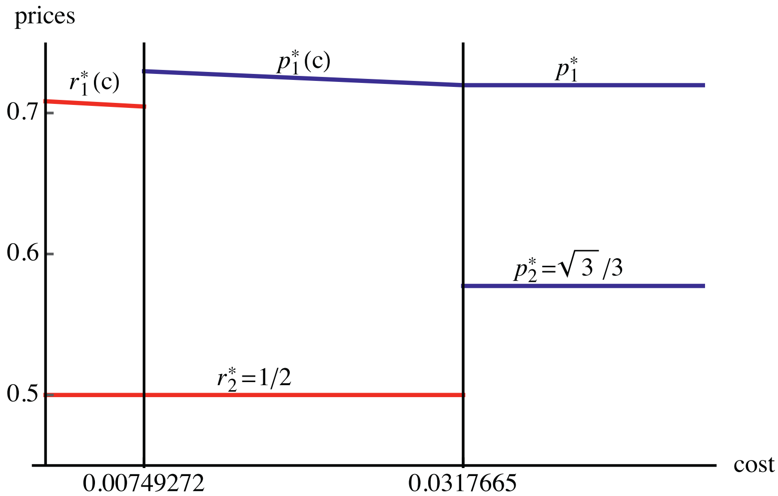

Fig. 5. The optimal reserve price and posted price as the auction cost varies. When the auction cost is smaller than 0.0075, the optimal mechanism sequence is auction with reserve price $r_1^*(c)$ and auction with reserve price 1/2. When the auction cost is between 0.0075 and 0.032, the optimal mechanism sequence is price $p_1^*(c)$ and auction with reserve price 1/2. When the auction cost is bigger than 0.032, the optimal mechanism sequence is price p_1^* in the first period and price $\sqrt{3}/3$ in the second period.

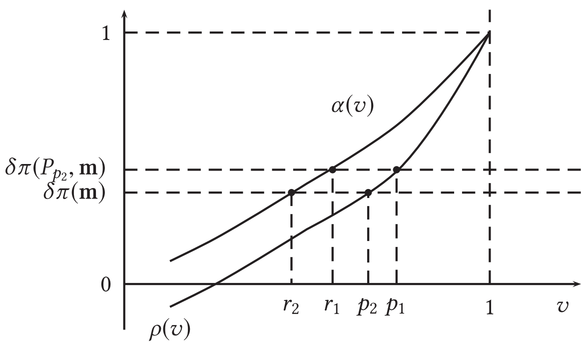

Fig. 6. $(A_{r_1}, P_{p_2}, \mathbf{m})$ when $r_1 \leq p_2$ cannot simultaneously dominate $(P_{p_2}, P_{p_2}, \mathbf{m})$ and $(A_{r_1}, A_{r_2}, \mathbf{m})$.

Given the optimal solution in period T, we can solve for the seller's problem in period $T-1$:

$$\max_{m_{T-1}} \quad \pi(m_{T-1}) + \delta k(m_{T-1})\pi_T^*(c),$$

where the maximized expected profit is the larger of the expected profit of running the optimal auction and that of posting the optimal price,

$$\pi_T^*(c) = \max\left\{R(A_{r_T^*}) - c, R(P_{p_T^*})\right\}. \tag{4}$$

If she chooses an auction A_r in period $T-1$, then her expected profit is

$$\pi(A_r, m_T^*(c)) = \delta\pi_T^*(c) + \int_r^1 [\alpha(v) - \delta\pi_T^*(c)]dF^n(v) - c,$$

where the first term is the net present value of the period T profit, and the second term is the revenue from holding the auction in period $T-1$, and c is the auction cost. Similarly, if she chooses a posted price P_p in period $T-1$, then her expected profit is

$$\pi(P_p, m_T^*(c)) = \delta\pi_T^*(c) + \int_p^1 [\rho(v) - \delta\pi_T^*(c)]dF^n(v).$$

Her optimal reserve price is $r_{T-1}^*(c) = \alpha^{-1}(\delta\pi_T^*(c))$ and her optimal posted price is $p_{T-1}^*(c) = \rho^{-1}(\delta\pi_T^*(c))$. The seller's optimal profit in period $T-1$ is

$$\pi_{T-1}^*(c) = \delta\pi_T^*(c) + \max\left\{\int_{r_{T-1}^*(c)}^1 [\alpha(v) - \delta\pi_T^*(c)]dF^n(v) - c, \int_{p_{T-1}^*(c)}^1 [\rho(v) - \delta\pi_T^*(c)]dF^n(v)\right\}.$$

There is a cutoff cost c_{T-1}^* such that the seller runs $A_{r_{T-1}^*(c)}$ if her cost is lower than it and runs $P_{p_{T-1}^*(c)}$ otherwise, where

$$c_{T-1}^* = \int_{\delta\pi_t^*(c_{T-1}^*)}^1 [x - \delta\pi_T^*(c_{T-1}^*)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right].$$

The entire optimal mechanism sequence can be solved by iterating this procedure over all periods $t \le T-1$. The optimal mechanism sequence is summarized in Lemma 2.

Lemma 2. *Suppose T is finite. For any $t \le T-1$, let $\pi_{t+1}^*(c)$ denote the optimal expected profit of a cost c seller in period $t+1$. Let $r_t^*(c) = \alpha^{-1}(\delta\pi_{t+1}^*(c))$ and $p_t^*(c) = \rho^{-1}(\delta\pi_{t+1}^*(c))$, and let c_t^* be the unique solution to*

$$c_t^* = \int_{\delta\pi_{t+1}^*(c_t^*)}^1 [x - \delta\pi_{t+1}^*(c_t^*)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right]. \tag{5}$$

A cost c seller's optimal mechanism $m_t^*(c)$ in period $t \le T-1$ is $A_{r_t^*(c)}$ if $c < c_t^*(c)$, is $P_{p_t^*(c)}$ if $c > c_t^*(c)$, and is $A_{r_t^*(c)} = P_{p_t^*(c)}$ if $c = c_t^*(c)$.

Proof of Lemma 2.. We need to show that Eq. (5) rearranged as below has a unique solution c_t^* for each t,

$$\gamma(c_t^*) \equiv c_t^* - \int_{\delta\pi_{t+1}^*(c_t^*)}^1 [x - \delta\pi_{t+1}^*(c_t^*)]d\left[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\right] = 0. \tag{6}$$

It suffices to show that $\gamma(c)$ is continuous and increasing in c, $\gamma(0) < 0$, and $\gamma(1) > 0$. Note that $\pi_{t+1}^*(c)$ is differentiable except at a finite number of points, and at the non-differentiable points, the left derive and right derivative exist, and they are monotonic. Hence, $\gamma(c)$ is differentiable except at a finite number of points. At differentiable points c,

$$\begin{aligned}
\gamma'(c) &= 1 + \delta\pi_{t+1}^{*'}(c)\int_{\delta\pi_{t+1}^*(c)}^1 d[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))] \\
&= 1 - \delta\pi_{t+1}^{*'}(c)[F^n(\alpha^{-1}(\delta\pi_{t+1}^*(c))) - F^n(\rho^{-1}(\delta\pi_{t+1}^*(c)))] \\
&= 1 - \delta\pi_{t+1}^{*'}(c)[k(r_t^*) - k(p_t^*)].
\end{aligned}$$

The maximal profit is

$$\pi_{t+1}^*(c) = \max\{R(A_{r_{t+1}^*}) - c + \delta k(r_{t+1}^*)\pi_{t+2}^*(c), R(P_{p_{t+1}^*}) + \delta k(p_{t+1}^*)\pi_{t+2}^*(c)\}.$$

Therefore, for any t,

$$\pi_{t+1}^{*'}(c) \ge -1 + \delta k(r_{t+1}^*)\pi_{t+2}^{*'}(c),$$

and

$$\pi_{t+2}^{*'}(c) \ge -1 + \delta k(r_{t+2}^*)\pi_{t+3}^{*'}(c),$$

and so on for $\pi_T^{*'}(c)$ through T. Altogether, the inequalities, coupled with the inequalities that $r_t^* > r_{t+1}^*$ for all t, imply that

$$\begin{aligned}\pi_{t+1}^{*'}(c) &\geq -[1 + \delta k(r_{t+1}^*) + \delta^2 k(r_{t+1}^*)k(r_{t+2}^*) + \cdots] \\ &\geq -[1 + \delta k(r_{t+1}^*) + \delta^2 k^2(r_{t+1}^*) + \cdots] \\ &= -\frac{1}{1 - \delta k(r_{t+1}^*)} \geq -\frac{1}{1 - \delta k(r_t^*)}.\end{aligned}$$

Therefore,

$$\gamma'(c) \geq 1 - \frac{\delta k(p_t^*) - \delta k(r_t^*)}{1 - \delta k(r_t^*)} = \frac{1 - \delta k(p_t^*)}{1 - \delta k(r_t^*)} > 0.$$

At the non-differentiable points, the steps above hold for left derivatives and right derivatives, respectively, so the proof shows that $\gamma(c)$ is continuous and increasing everywhere.

Finally, since $\int_{\delta\pi_{t+1}^*(c)}^1 [x - \delta\pi_{t+1}^*(c)]d\big[F^n(\alpha^{-1}(x)) - F^n(\rho^{-1}(x))\big]$ is the revenue difference between the optimal auction and the optimal price in a period, it is between 0 and 1 for any c, and $\gamma(0) < 0$ and $\gamma(1) > 0$ follow directly. □

The optimal profit in period T is determined by Eq. (4), and the optimal profit in period $t \leq T - 1$ is

$$\pi_t^*(c) = \max\left\{R(A_{r_t^*(c)}) - c + k(r_t^*(c))\delta\pi_{t+1}^*(c),\ R(P_{p_t^*(c)}) + k(p_t^*(c))\delta\pi_{t+1}^*(c)\right\} \tag{7}$$

The optimal mechanism sequence is thus completely characterized by the two lemmas and each period's optimal profit function.

Lemma 3. *Suppose T is finite. The optimal mechanism sequence $(m_1^*(c), \cdots, m_T^*(c))$ is characterized by Lemmas 1 and 2, with $\pi_t^*(c)$ defined by Eqs. (4) and (7), and c_t^* determined by Eq. (5).*

Lemma 3 completely characterizes the seller's problem for any cost c seller, but the solution is not very informative. We only know that there is a cutoff cost c_t^* in each period t such that a seller chooses a reserve price auction if her cost is smaller than c_t^* and posts a price if her cost is bigger than c_t^*. In other words, all we know so far is that in each period if the auction cost is low, use an auction, and if the auction cost is high, post a price. A little bit more work gives us the final neater result. We can show that the cutoff costs increase over the periods. In other words, it is more and more likely a seller will run an auction in a later period.

Proof of Theorem 1.. Eq. (5) determines the cutoff cost c_t^* in each period. In Eq. (5), the cutoff cost c_t^* is decreasing in $\delta\pi_{t+1}^*(c)$. Since $\pi_{t+1}^*(c) > \pi_t^*(c)$ for any t, $c_t^* < c_{t+1}^*$ for any t. □

It is more profitable paying the auction cost later than earlier. In the earlier periods, there are still many more periods left and many opportunities to sell the good, and it is not worth paying an auction cost, because the good has a retention value. However, as time passes on and the sale opportunities diminish, the auction cost, though constant in absolute terms, is deemed more attractive relative to the risk of not selling the good.

5.1. Sub-optimality of auctions-and-prices sequences

As a corollary of Theorem 1, it is impossible for any mechanism sequence that has an auction before a posted price to be optimal. In other words, all mechanism sequences consisting of an auction and a posted price following it are always dominated strictly by at least one alternative mechanism sequence–either dynamic pricing, sequential auctions, or posted prices followed by auctions.

Corollary 1. *Suppose T is finite. An auctions-then-prices sequence is never optimal when buyers are short-lived. Consequently, any mechanism sequence with an auctions-then-prices sequence is not optimal.*

Here I present a proof independent of the previous proofs. I directly construct the mechanisms that dominate the profit-maximizing auction-price sequence.

An Alternative Proof of Corollary 1.. Because the mechanisms chosen before period t do not affect the optimal mechanism sequence after period t, without loss of generality, it suffices to show that the mechanism sequence of an auction in the first period and then a posted price in the second period is never optimal. Suppose the seller runs the mechanism sequence $\mathbf{m}$ in periods 3 through T and generates expected profit $\pi(\mathbf{m})$. It suffices to show that the optimal sequence $(A_{r_1}, P_{p_2}, \mathbf{m})$ is dominated by at least one other mechanism sequence not consisting of the auction-price sequence.

Suppose that $(A_{r_1}, P_{p_2}, \mathbf{m})$ is optimal and $r_1 > p_2$. The optimal r_1 and p_2 are determined respectively by $\rho(p_2) = \delta\pi(\mathbf{m})$ and $\alpha(r_1) = \delta\pi(P_{p_2}, \mathbf{m}) = \delta R\big(P_{p_2}\big) + \delta^2 k(p_2)\pi(\mathbf{m})$.$(A_{r_1}, P_{p_2}, \mathbf{m})$ dominates both $(A_{r_1}, A_{p_2}, \mathbf{m})$ and $(P_{r_1}, P_{p_2}, \mathbf{m})$. That is, $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(A_{r_1}, A_{p_2}, \mathbf{m})$ and $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(P_{r_1}, P_{p_2}, \mathbf{m})$, where

$$\begin{aligned}\pi(A_{r_1}, P_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1)R(P_{p_2}) + \delta^2 k(r_1)k(p_2)\pi(\mathbf{m}), \\ \pi(A_{r_1}, A_{p_2}, \mathbf{m}) &= R(A_{r_1}) - c + \delta k(r_1)[R(A_{p_2}) - c] + \delta^2 k(r_1)k(p_2)\pi(\mathbf{m}),\end{aligned}$$

$$\pi(P_{r_1}, P_{p_2}, \mathbf{m}) = R(P_{r_1}) + \delta k(r_1) R(P_{p_2}) + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}).$$

The two inequalities become

$$R(P_{p_2}) - [R(A_{p_2}) - c] \geq 0,$$
$$[R(A_{r_1}) - c] - R(P_{r_1}) \geq 0.$$

Adding the two inequalities up,

$$R(A_{r_1}) - R(P_{r_1}) \geq c \geq R(P_{p_2}) - R(A_{p_2}).$$

Since $R(A_r) - R(P_r) = \int_r^1 [\alpha(v) - \rho(v)] dF^n(v)$ is strictly decreasing in r, and since $r_1 > p_2$,

$$R(A_{r_1}) - R(P_{r_1}) < R(P_{p_2}) - R(A_{p_2}),$$

a contradiction with the inequality above and the premise that $\left(A_{r_1}, P_{p_2}, \mathbf{m}\right)$ is optimal.

Suppose that $\left(A_{r_1}, P_{p_2}, \mathbf{m}\right)$ is optimal and $r_1 \leq p_2$. It should dominate all other alternative mechanism sequences, in particular both $(P_{p_2}, P_{p_2}, \mathbf{m})$ and $(A_{r_1}, A_{r_2}, \mathbf{m})$ where $r_2 = \alpha^{-1}(\delta\pi(\mathbf{m}))$. That is, $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(P_{p_2}, P_{p_2}, \mathbf{m})$ and $\pi(A_{r_1}, P_{p_2}, \mathbf{m}) \geq \pi(A_{r_1}, A_{r_2}, \mathbf{m})$, where

$$\pi(A_{r_1}, P_{p_2}, \mathbf{m}) = R(A_{r_1}) - c + \delta k(r_1) R(P_{p_2}) + \delta^2 k(r_1) k(p_2) \pi(\mathbf{m}),$$
$$\pi(P_{p_2}, P_{p_2}, \mathbf{m}) = R(P_{p_2}) + \delta k(p_2) R(P_{p_2}) + \delta^2 k(p_2) k(p_2) \pi(\mathbf{m}),$$
$$\pi(A_{r_1}, A_{r_2}, \mathbf{m}) = R(A_{r_1}) - c + \delta k(r_1)[R(A_{r_2}) - c] + \delta^2 k(p_1) k(r_2) \pi(\mathbf{m}).$$

We can plug in and rewrite the two inequalities as

$$R(A_{r_1}) - c - R(P_{p_2}) - \delta[k(p_2) - k(r_1)]\pi(R_{p_2}, \mathbf{m}) \geq 0,$$
$$R(P_{p_2}) - R(A_{r_2}) + c - \delta[k(p_2) - k(r_2)]\pi(\mathbf{m}) \geq 0.$$

Summing the two inequalities,

$$R(A_{r_1}) - R(A_{r_2}) - \delta[k(p_2) - k(r_1)]\pi(R_{p_2}, \mathbf{m}) - \delta[k(p_2) - k(r_2)]\pi(\mathbf{m}) \geq 0. \tag{8}$$

r_1 and r_2 satisfy $\alpha(r_1) = \delta\pi(P_{p_2}, \mathbf{m})$ and $\alpha(r_2) = \delta\pi(\mathbf{m})$. By the optimality of $(A_{r_1}, P_{p_2}, \mathbf{m})$, $\pi(P_{p_2}, \mathbf{m}) > \pi(\mathbf{m})$. As illustrated by Fig. 2, $r_T^* \leq r_2 < r_1$, so $R(A_{r_1}) < R(A_{r_2})$. Coupled with the inequality $r_2 < r_1 \leq p_2$,

$$[R(A_{r_1}) - R(A_{r_2})] - \delta[k(p_2) - k(r_1)]\pi(R_{p_2}, \mathbf{m}) - \delta[k(p_2) - k(r_2)]\pi(\mathbf{m}) < 0,$$

contradicting inequality (8) above. □

5.2. Extensions

The main result holds when the deadline is stochastic, the seller is increasingly impatient, auction costs are decreasing, buyer arrival is stochastic, buyers have outside options, buyers pay bidding costs, there is a separate market for prices and auctions, the seller is procuring contracts, and/or the seller is selling multiple goods sequentially. Appendix B presents the extensions that are straightforward. Here I include two extensions in which the continuation value is decreasing.

5.2.1. Increasingly impatient seller

Even when the seller survives the market less likely over time (μ_t decreases) or becomes increasingly impatient (δ_t decreases), the main results are unchanged. Because the assertion that discounted payoff in a period is still greater than that in a later period, $\mu_t \delta_t \pi_t^*(c) > \mu_{t+1} \delta_t \pi_{t+1}^*(c)$ for any c, the result that the optimal cutoff c_t^* increases over time is unaltered.

5.2.2. Decreasing auction costs

The result continues to hold even when the auction cost is not constant. Instead, it either declines over time, or declines whenever the seller uses an auction an additional time. To understand the result, remember the key trade-off between an auction and a posted price. The revenue advantage of an auction over a posted price increases in later periods, and the opportunity cost of selling a good by an auction over that by a posted price decreases in later periods. If the originally constant auction cost becomes smaller in later periods, the result that an auction is more profitable to be used in a later period than in an earlier period is reinforced.

6. Long-lived buyers

Thus far I have assumed that the buyers are short-lived. In this section, I consider the environment in which the buyers are long-lived (and the rest of the setting stays the same). First, I consider a dynamic programming approach to solve the general problem and state its limitations. Then I solve the two-period problem with long-lived buyers being either myopic or forward-looking. The main result remains: No auction-price sequence is optimal.

6.1. A dynamic programming approach

6.1.1. Short-lived buyers

Thanks to the results in Crémer et al. (2007) and Lee and Li (2020), we can think of the monopolist's dynamic sales problem with short-lived buyers as an optimal search problem à la Weitzman (1979). Weitzman (1979) considers a problem in which Pandora incurs search costs to sequentially open boxes with uncertain returns to maximize her expected payoff from choosing the box with the highest realized return. He shows that each box can be assigned a worth that only depends on the box's own characteristics (namely, the distribution of return of the box), and the optimal search procedure is to start searching from the box of the highest worth and stop whenever the realized return exceeds the highest worth of the boxes not yet opened. Crémer et al. (2007) show that any incentive feasible mechanism—a mechanism that has a perfect Bayesian equilibrium in which every invited bidder participates and is truthful—can be cast as an optimal search mechanism that uses the symmetric-information optimal search procedure of Weitzman (1979) relative to virtual utility functions (instead of actual values in Weitzman (1979)). Specifically, in the current setting, costly running an auction in a period is like spending a search cost c and opening a Pandora box containing n buyers with virtual utilitys $\alpha(v)$, so the worth of the box is the highest virtual utility $\alpha\big(\max_{i=1,\cdots,n} v_i\big)$ (Lee and Li, 2020). Costlessly posting a price is like spending no search cost and opening a box containing one buyer with virtual utility $\rho(v)$, because the transaction is complete if and only if the buyer with the highest value is willing to buy. The problem can be formulated as the following dynamic programming problem.[9] Let z denote the highest virtual value among the buyers at the end of a period. The seller can obtain the fallback revenue from the buyer, or continue to search by using an auction or using a price, so the dynamic programming problem for $t < T$ can be written as

$$J_t(z) = \max\{z, \delta[\mathbb{E}_x J_{t+1}(\alpha(x)) - c], \delta\mathbb{E}_x J_{t+1}(\rho(x))\},$$

where $x \sim F^n$, and $J_T(z) = \max\{z, 0\}$ and $J_{T+1}(z) = 0$.

Hence, in every period, there is a threshold virtual vale z such that the seller is indifferent between obtaining the fallback revenue z and continuing to search. the threshold in the last period is $z_T = 0$, and in every period $t < T$, the threshold is

$$z_t = \delta \max\{\mathbb{E}_x J_{t+1}(\alpha(x)) - c, \mathbb{E}_x J_{t+1}(\rho(x))\}.$$

In every period t, the seller is choosing between the costly auction with reserve price $\alpha^{-1}(z_t)$ and the price $\rho^{-1}(z_t)$. Our previous analysis shows that z_t is decreasing over time, and furthermore, as z_t decreases, the revenue difference between the costly auction and the price increases, so the costly auction is a more desirable choice.

6.1.2. Long-lived buyers

First, consider the dynamic programming approach. In every period t, the seller can choose to search n value $v \sim F$ buyers with a cost c, or one value $v \sim F^n$ buyer with no cost. These buyers will stay in the game. The seller's dynamic programming problem in period $t < T$ becomes

$$J_t(z) = \max\{z, \delta[\mathbb{E}_x J_{t+1}(\max\{z, \alpha(x)\}) - c], \delta\mathbb{E}_x J_{t+1}(\max\{z, \rho(x)\})\},$$

and $J_T(z) = \max\{z, 0\}$ and $J_{T+1}(z) = 0$. In every period $t < T$, the seller is indifferent between obtaining the fallback revenue z_A and choosing n buyers, where z_A satisfies

$$z_A = \delta\left[\int_0^1 \max\{z_A, \alpha(x)\} dF^n(x) - c\right],$$

and analogously, the seller is indifferent between obtaining the fallback revenue z_P and choosing n buyers, where z_P satisfies

$$z_P = \delta\left[\int_0^1 \max\{z_P, \rho(x)\} dF^n(x)\right].$$

The values of the "boxes" are determined by z_A and z_P, respectively, so the seller will choose n buyers or the one buyer accordingly, and since the choices are the same in different periods, the seller will choose the same "box" every period. Hence, the thresholds are $z_t = \max\{z_A, z_P\}$ for $t < T$, and $z_T = 0$.

However, to obtain the optimal profit, second-price auctions need to be run in *every* period (see Section 4 of Lee and Li (2020) for details). Namely, if $z_A \geq z_P$, the optimal sequence of mechanisms is second-price auctions that result in buyers with values greater than $\alpha^{-1}(z_t)$ participating in the auction in every period, where the reserve prices are constant from periods 1 to $T-1$, and drop to the Myerson reserve price $\alpha^{-1}(0)$ in period T. If $z_A < z_P$, one additional buyer is invited in every period, and the optimal sequence of mechanisms is essentially a sequence of constant prices before period T, and a (costless) fire sale auction in period T in which all buyers from previous periods can participate and may obtain the object with positive probabilities, because $\rho^{-1}(0) < \rho^{-1}(z_P)$.

[9] I thank a referee for this great insight.

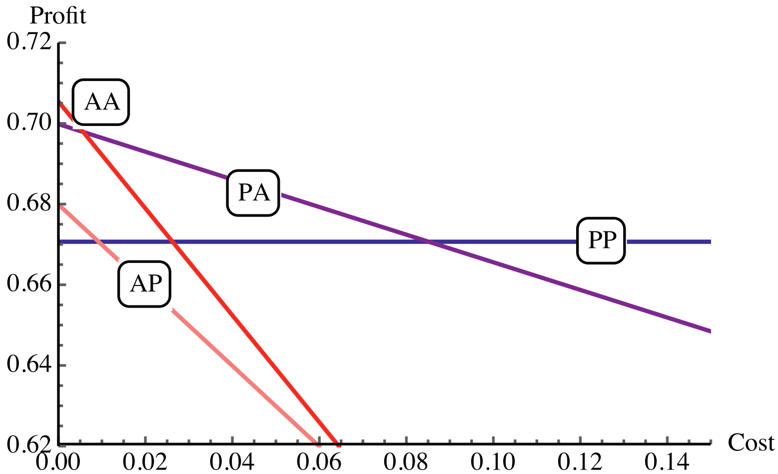

Fig. 7. The optimal profits of different mechanism sequences for different auction costs when buyers are long-lived and myopic. When the auction cost is smaller than 0.007, the optimal mechanism sequence is auction-auction. When the auction cost is between 0.007 and 0.08, the optimal mechanism sequence is price-auction. When the auction cost is bigger than 0.08, the optimal mechanism sequence is price-price.

Therefore, the dynamic programming approach cannot be applied to solve for the finite-horizon problem with long-lived buyers in which the seller chooses between a costly auction and a costless price every period. There are three problems associated with transforming the current problem with long-lived buyers to a costly search problem. First, the cost on the mechanism cannot be directly translated to be a cost on searching for buyers in the setting with long-lived buyers. This is not a problem in the setting with short-lived buyers, because the cost on the static mechanism and the buyers participating in that mechanism are relevant for one period only, but the long-lived buyers who participate in a mechanism will continue to participate in all subsequent mechanisms. The cost of the mechanism in a period can be directly interpreted as the cost of inviting the buyers in that period in the short-lived setting but not in the long-lived setting. Second, the fallback revenue cannot be guaranteed in the long-lived setting when auctions and prices are chosen. The fallback revenue can be obtained when the seller uses a second-price auction, but in the current setting, when a price is used in a period, that fallback revenue cannot be guaranteed. Third, the posted price's marginal revenue function ρ is difficult to interpret. The revenue of a posted price mechanism with symmetric buyers can be succinctly represented with the help of ρ, but when there are asymmetric buyers across periods participating in the mechanism in the same period, the function loses an economic interpretation and its usefulness.

Nonetheless, this dynamic programming approach helps obtain some useful results. First, it can be deduced that whenever $z_A \geq z_P$, the optimal sequence of mechanisms is the sequence of costly auctions derived above, because even allowing some costless auctions that generate higher optimal revenues than costless prices does not generate a higher revenue than the costly auctions specified. Second, when the problem takes an infinite horizon, this approach is valid, because the optimal sequence of mechanisms is either a sequence of constant prices or a sequence of auctions with constant reserve prices, and the solution just obtained is the solution with long-lived buyers, and also coincides with the solution with short-lived buyers. Third, even in this relaxed setting, the optimal mechanism sequence only takes the form of auctions or prices then auctions; no auctions-prices sequence is optimal.

6.2. Two-period problems with long-lived buyers

6.2.1. Long-lived myopic buyers

Now we consider long-lived buyers who are myopic. They buy as if they only live for one period. The proposition below shows the general result when there are two periods and an arbitrary number of buyers. The proofs leverage the alternative proof of Corollary 1.

Proposition 2. *Suppose there are two periods and buyers are long-lived and myopic. An auction-price sequence is never optimal. The optimal sequence is an auction-auction sequence when the auction cost is sufficiently small, is a price-price sequence when the auction cost is sufficiently large; and is a price-auction sequence otherwise.*

Fig. 7 shows that the optimal revenues of different mechanism sequences when there are two long-lived myopic buyers with uniform [0,1] distributions in each period and $\delta = 0.8$.

6.2.2. Long-lived forward-looking buyers

Now we consider the case with forward-looking buyers. Assume that the buyers born in the first period continue to be around in the second period and have a chance to buy the good in the second period if the good has not been sold. Fur-

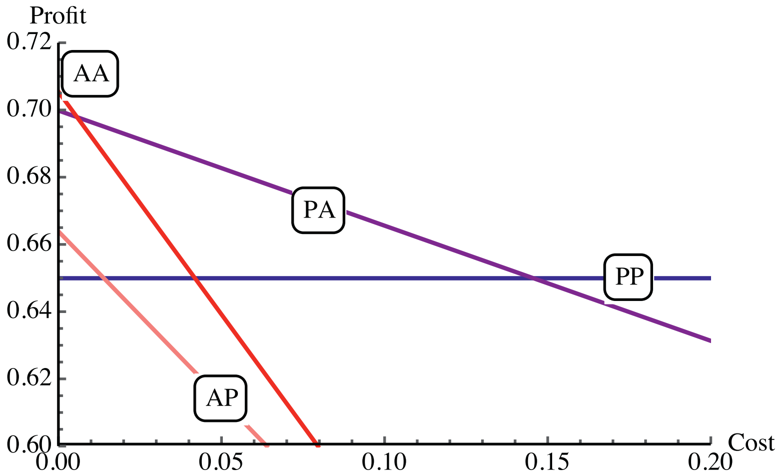

Fig. 8. The optimal profits of different mechanism sequences for different auction costs when buyers are long-lived and forward-looking. When the auction cost is smaller than 0.007, the optimal mechanism sequence is auction-auction. When the auction cost is between 0.007 and 0.15, the optimal mechanism sequence is price-auction. When the auction cost is bigger than 0.15, the optimal mechanism sequence is price-price.

thermore, the seller and the buyers discount by $\delta < 1$ (if the seller does not discount, the optimal mechanism is trivial: wait until the last period to run the static optimal auction or post the static optimal price). Because buyers are forward-looking, we can no longer solve the seller's problem backwards like we have done when buyers are short-lived. The mechanisms and prices chosen in the latter periods affect buyers' behavior in earlier periods and in turn the seller's profit in the earlier periods. The optimal prices are characterized by a system of equations. In the appendix, we solve one by one the optimal price-price, price-auction, auction-auction, and auction-price sequences when there are two buyers.

Despite of more complication, the main result remains: the optimal mechanism sequence is auction-auction when the auction cost is sufficiently low, price-auction when the auction cost is intermediate, and price-price when the auction cost is sufficiently high, and an auction-price sequence is never optimal.

Proposition 3. *Suppose there are two periods and the buyers are long-lived and forward-looking. An auction-price sequence is never optimal. The optimal sequence is an auction-auction sequence when the auction cost is sufficiently small, is a price-price sequence when the auction cost is sufficiently large; and is a price-auction sequence otherwise.*

Fig. 8 shows the optimal revenues of different mechanism sequences when two long-lived, forward-looking buyers arrive in each of the two periods with values drawn from the uniform [0,1] distribution, and the common discount factor is $\delta = 0.8$. Note that for only very small auction costs ($c < 0.007$) the auction-auction sequence is optimal, and for only very large auction cost ($c > 0.15$) the price-price sequence is optimal. For the most economically relevant range, the price-auction sequence is optimal.

7. Conclusion

This paper studies a monopolist's expected profit maximizing sequence of costless posted prices and costly auctions when buyers arrive to the market over time. Most interestingly, we find that in a variety of settings, posting a price before auctioning is optimal. An optimal auction has a higher sale probability than an optimal posted price, so the use of an auction in an earlier period is associated with a higher probability of giving up the good. An auction incurs not only a constant fixed operational cost, but also an endogenously higher opportunity cost associated with the retention value of the good. A prices-then-auctions sequence is more desirable than an auctions-then-prices sequence because the endogenous opportunity cost of selling through an auction is decreasing over time. On eBay in particular, a Buy It Now option that allows a buyer to snatch a good for a fixed price before the seller starts an auction resembles the optimal prices-then-auctions mechanism sequence presented in this paper. Finally, a constant price can be optimal when the seller does not have a deadline to sell the good. When the market transaction speeds up, it is more likely to use posted prices, as we have observed on eBay.

The following extensions are interesting and worthwhile to be explored. First, the seller may have multiple copies of the identical goods. We have explored the possibility that the seller sells the good sequentially, but it is worthwhile to consider other forms of auctions in which the seller can sell more than one good at a time. The dynamic programming problem carries one more state variable besides the seller's age: the number of remaining objects she has. Given the complexity of the problem with even one good, the multiple-goods extension should be treated as a separate pursuit complementary to the current one.

Second, the seller has been assumed to commit to a particular sequence of mechanisms and the commitment as well as the mechanism are public knowledge. This assumption can be relaxed. Skreta (2006) shows that simple prices can be

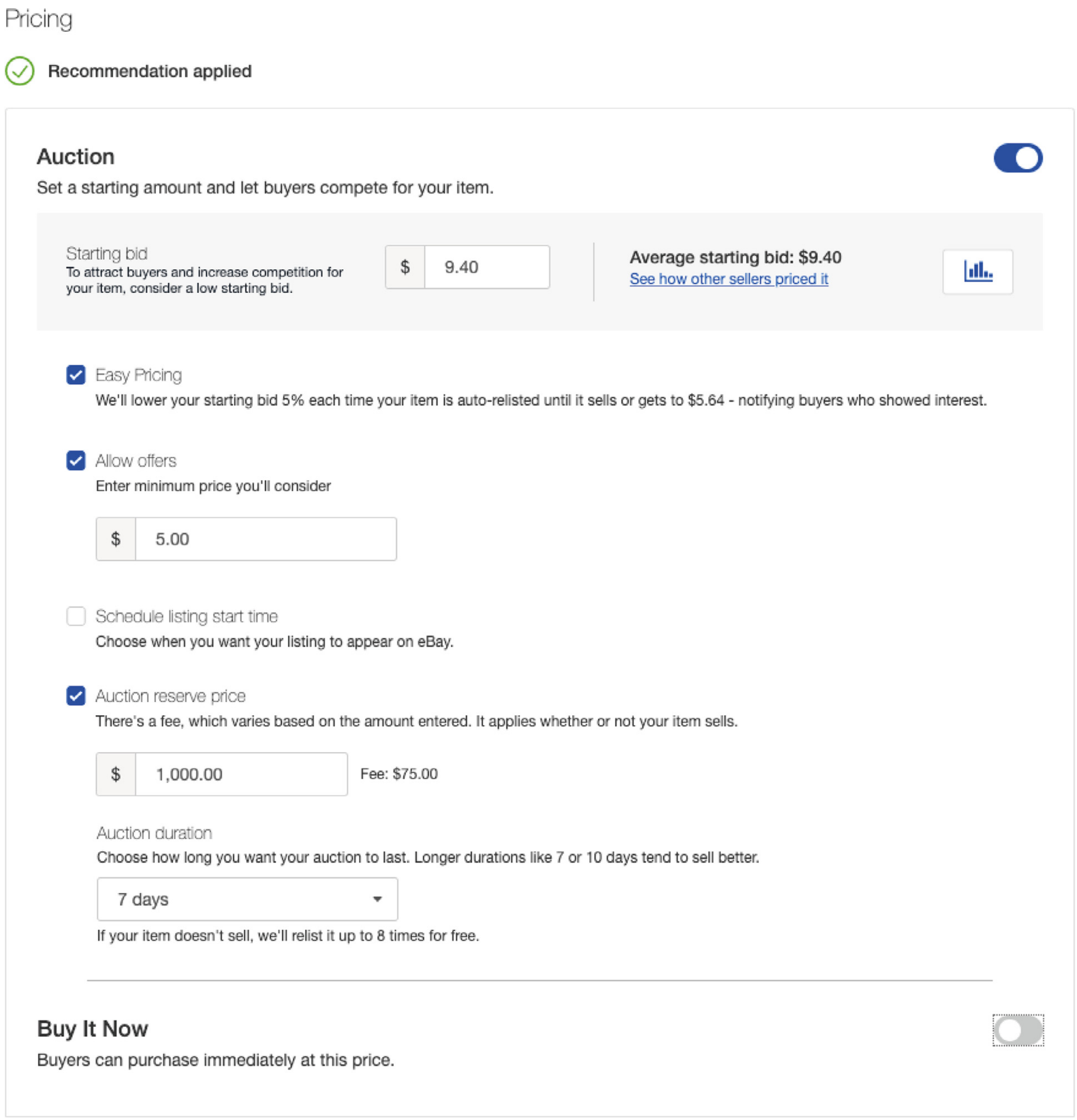

Fig. A1. A screenshot of options when listing an auction on eBay. Figs. A.2 and A.3 in the appendix show screenshots of options when listing a Buy It Now price and a Buy It Now auction, respectively. A seller is allowed to automatically lower bids, allow offers below starting bids, schedule a listing start time, and set a reserve price with a considerable fee.

optimal when the seller does not have commitment power. Dilmé and Li (2019) consider the equilibrium price paths of a non-committal monopolist in a revenue management setting.

Appendix A. Screenshots of options when listing on eBay

Fig. A.1 shows a screenshot of options when listing an auction on eBay. Fig. A.2 shows a screenshot of options when listing a Buy It Now. Fig. A.3 shows a screenshot of options when listing a hybrid auction (auction and Buy It Now).

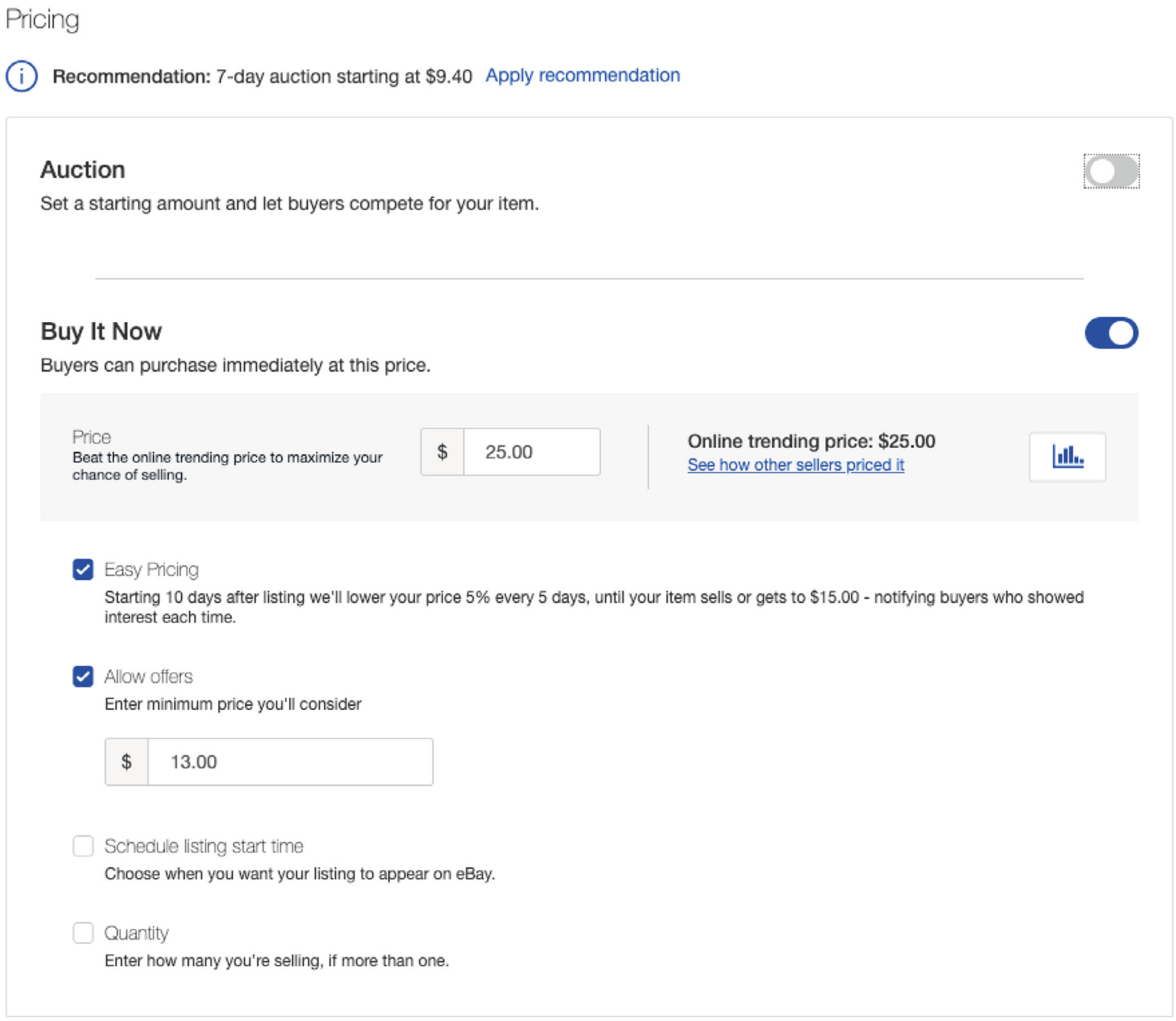

Fig. A2. A screenshot of options when listing a Buy It Now on eBay.

Appendix B. Finite-horizon extensions with short-lived buyers

The previous section shows two main results: the optimality of the prices-then-auctions sequence in Theorem 1 and the sub-optimality of the auctions-then-prices sequence in Corollary 1. This section extends the benchmark setting and shows that both results continue to hold. We show the robustness of the result when the seller has stochastic sale deadline, increasing impatience, or declining auction cost, when the buyers arrive stochastically, have outside options, incur a bidding cost, when markets are separate for auctions and posted prices, when the mechanism designer is procuring a contract from potential contractors rather than selling a good to buyers, and when multiple goods are sold sequentially.

B1. Stochastic sale deadline

Suppose the seller survives the market with probability $\mu < 1$ to the next period. The stochastic deadline may result from the seller's uncertainty about when or whether her good will be out of favor, or when it may be perished or banned to be sold. It has a similar effect as time discounting. The expected payoff of the future periods becomes $\mu\delta\pi_{T+1}$; by redefining $\delta' = \mu\delta$, the problem is the same as before.

B2. Stochastic buyer arrival

If in each period, the buyer arrival process is stochastic instead of deterministic, the results do not change. Suppose that the probability that the number of buyers is n is $q(n)$, then the probability of selling is $s(p) = 1 - \sum q(n)F^n(p)$. The optimal

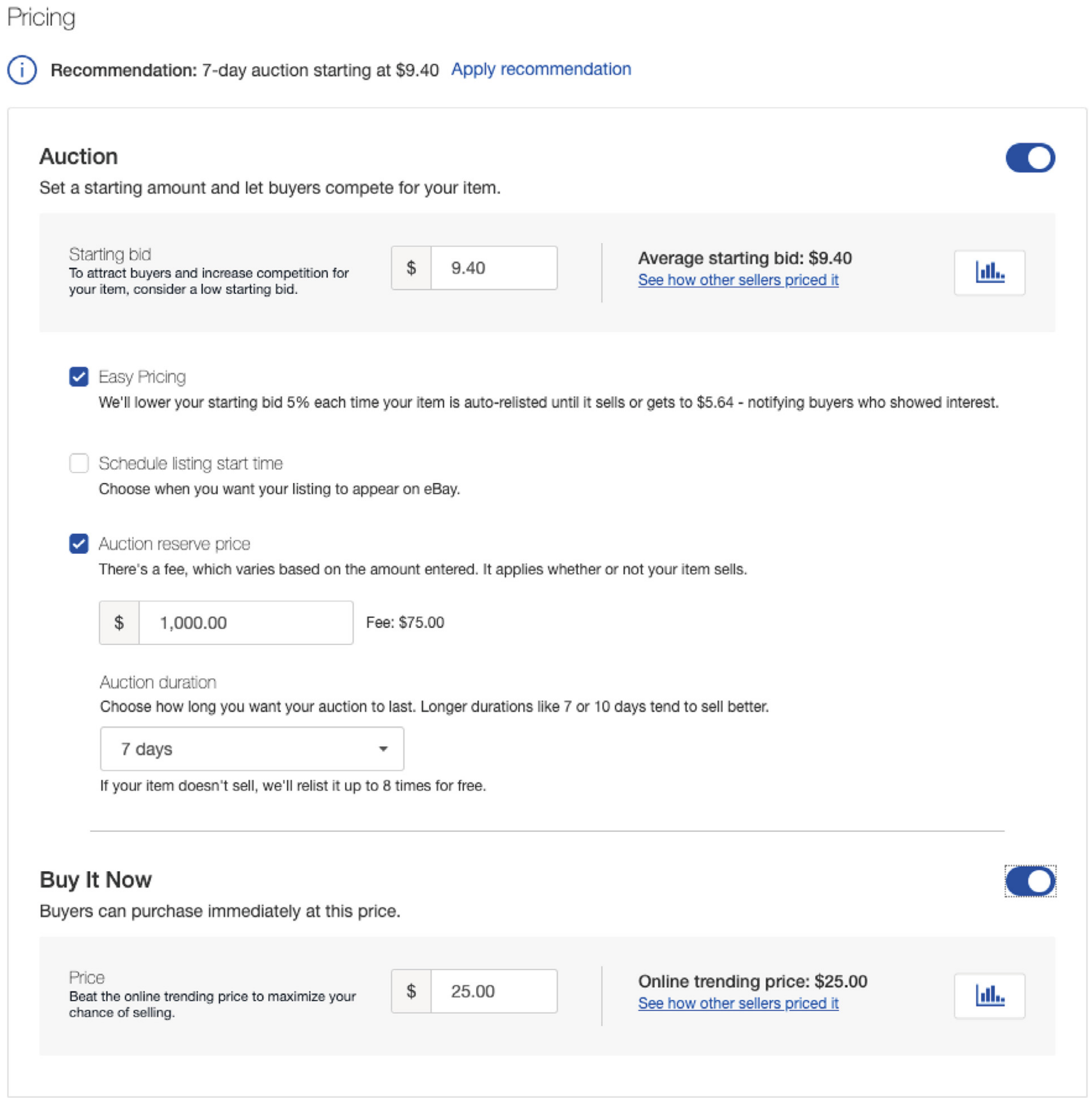

Fig. A3. A screenshot of options when listing a hybrid auction on eBay.

posted price is then determined by

$$\rho(p_t^*) = p_t^* - \frac{1 - F^n(v)}{[F^n(v)]'} = p_t^* + \frac{s(p_t^*)}{s'(p_t^*)} = \delta\pi_{t+1}^*(c).$$

A posted price's expected revenue is $R\left(P_p\right) = ps(p)$ and an auction's expected revenue is

$$R(A_r) = \int_r^1 \alpha(v) d\left[\sum_n q(n) F^n(v)\right].$$

Since the determination of c_t^* is not changed from Eq. (5), the results from the previous section on the form of mechanism sequence carry over.

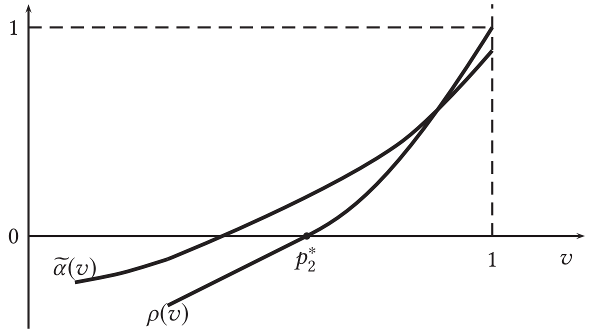

Fig. A4. The effective marginal revenue curve when buyers pay a bidding cost c_b. The auction's marginal revenue curve shifts down to $\widetilde{\alpha}(v)$.

B3. Buyers with outside options

Suppose that buyers have other options to buy the good after the current period (e.g. Satterthwaite and Shneyerov, 2007; Satterthwaite and Shneyerov, 2008). The buyers' willingnesses-to-pay in the current period that determine their bids in the auction and their purchasing decision in the posted price are depressed. The seller solves the problem with respect to the buyers' willingness-to-pay instead of their values. As long as the willingness-to-pay $w(v)$ is a concave transformation of the value v, the new marginal revenue curves become $\widetilde{\alpha}(v) = v - \frac{1-F(v)}{f(v)} w'(v)$ and $\widetilde{\rho}(v) = v - \frac{1-F^n(v)}{F^n(v)} w'(v)$. $\widetilde{\alpha}(v)$ and $\widetilde{\rho}(v)$ are still increasing and the main results continue to hold.

B4. Buyers with bidding costs

Suppose that in addition to the seller who incurs the auction cost c, each buyer also incurs a bidding cost $c_b \geq 0$ when bidding. (Similar to the seller's auction cost, the buyer's cost is not necessarily an actual physical cost he pays, but could also be a mental cost or an opportunity cost of participating in the current auction.) The seller who runs an auction A_r is effectively setting a higher reserve price than the actual reserve price r because only the buyers with values above some $\widetilde{r} > r$ bid. $\widetilde{r}$ satisfies $F^{n-1}(\widetilde{r})(\widetilde{r} - r) = c_b$. The expected profit of A_r followed by a mechanism sequence which generates profit π is

$$\int_{\widetilde{r}}^{1} \left[\alpha(v) - \frac{c_b}{F^{n-1}(v)} - \delta\pi\right] dF^n(v) + \delta\pi .$$

The bidding cost c_b generates a new effective marginal revenue curve $\widetilde{\alpha}(v) = \alpha(v) - \frac{c_b}{F^{n-1}(v)}$, lower than the original marginal revenue curve $\alpha(v)$. The bidding cost c_b lowers the seller's revenue by $c_b/F^{n-1}(v)$, even more than c_b. The optimal effective reserve price $\widetilde{r}$ is determined by $\widetilde{\alpha}(\widetilde{r}) = \delta\pi$ (the actual reserve price is $r = \widetilde{r} - c_b/F^{n-1}(\widetilde{r})$). Fig. A4 depicts the effective marginal revenues and the optimal expected revenues.

The reformulation of the problem with the effective marginal revenue curves easily shows that Theorem 1 continues to hold.

B5. Separate markets for prices and auctions

Suppose that the auction market and the posted price market are separate: n_A buyers will show up for an auction and n_P buyers will show up for a posted price. The revenue difference between running an optimal auction and an optimal price becomes

$$R(A_{r_t^*}, \mathbf{m}) - R(P_{p_t^*}, \mathbf{m}) = \int_{\delta\pi(\mathbf{m})}^{1} [x - \delta\pi(\mathbf{m})] d\left[F^{n_A}(\alpha^{-1}(x)) - F^{n_P}(\rho^{-1}(x))\right] \equiv \Delta(\delta\pi(\mathbf{m})).$$

$\Delta(0) > 0$ and $\Delta(1) = 0$, and

$$\Delta'(\delta\pi(\mathbf{m})) = -\delta \int_{\pi}^{1} d\left[F^{n_A}(\alpha^{-1}(x)) - F^{n_P}(\rho^{-1}(x))\right] = \delta\left[F^{n_A}(\alpha^{-1}(\pi)) - F^{n_P}(\rho^{-1}(\pi))\right].$$

The characterization of the optimal mechanism sequence remains.

In general, if the buyers arrive stochastically, let $q_A(n)$ denote the probability that n buyers arrive when an auction is run, and $q_P(n)$ the probability that n buyers arrive when a posted price is run. The difference between the optimal revenues is then

$$R(A_{r_t^*}, \mathbf{m}) - R(P_{p_t^*}, \mathbf{m}) = \int_{\delta\pi(\mathbf{m})}^{1} [x - \delta\pi(\mathbf{m})] d\left[\sum q_A(n) F^n(\alpha^{-1}(x)) - \sum q_P(n) F^n(\rho^{-1}(x))\right].$$

We say that an auction faces a higher demand than a posted price if $\sum_{i=1}^{n} q_A(i) \leq \sum_{i=1}^{n} q_P(i)$ for all n. Theorem 1 continues to hold.

B6. Procurement contracts

Throughout, we have considered the setting in which a monopolist is selling a good. We now turn our attention and consider the problem of a procurement contract in which the monopolist is looking for contracts to complete a project with the lowest cost. We show that the optimal mechanism sequence takes the same form. McAfee and McMillan (1988) consider the procurement problem when the communication cost is high and similar to our argument, find that the principal may search sequentially for contractors if the communication is too costly. A principal who needs a cost of 1 to complete the task, finds a contractor to finish a task for a minimum compensation. Suppose n contractors whose true costs c for the project are independently and identically distributed according to G arrive each period; assume G has increasing hazard rate G/g. The principal's choice is the same as the seller in the previous sections. Her auction's expected profit with a requirement of a maximum bid r is

$$\pi(A_r) = \int_0^r \left[1 - c - \frac{G(c)}{g(c)}\right] dG^n(c) - c.$$

The sale probability becomes $s(r) = G^n(r)$. Posted price P_p yields expected revenue $(1-p)G^n(p)$. The optimal reserve price and the optimal posted price are determined by

$$r_t^* + G(r_t^*)/g(r_t^*) = p_t^* + G^n(p_t^*)/(G^n(p_t^*)') = \delta\pi_{t+1}^*(c). \tag{9}$$

In the static setting, the optimal auction yields more expected revenue than the optimal posted price. In a finite horizon, the counterpart of Eq. (5) is

$$c_t^* = R(A_{r_t^*(c_t^*)}) - R(P_{p_t^*(c_t^*)}) - [k(p_t^*(c_t)) - k(r_t^*(c_t))]\delta\pi_{t+1}^*(c_t). \tag{10}$$

Therefore, the optimal sequence of mechanisms in the procurement problem depends on the auction cost in the same way as the seller's problem in the benchmark setting.

B7. Multiple goods

Suppose that a seller has multiple not necessarily identical goods (e.g. movie tickets of different seats) and must sell them sequentially by a certain deadline. The revenue from selling these goods is the sum of the individual revenues. Therefore, each good is sold by the optimal sequence of mechanisms as characterized. The revenue from the remaining good enters as an additional continuation payoff in the seller's sale decision. The existence of additional subsequently sold goods lowers the optimal reserve price or posted price of each good.

Appendix C. Infinite-horizon problem

Unlike goods with expirations such as airline tickets and hotel rooms or seasonal clothes that might fall out of favor in three months, many goods such as cellphones, stamps and books do not lose value and do not have fixed deadlines to be sold, but nonetheless, the seller has incentive to sell the good and realize the payment as soon as possible. In this section, the optimal sequence of mechanisms in the infinite-horizon is characterized. The solution turns out to be relatively simple and straightforward: the optimal mechanism sequence is an infinite repetition of the same mechanism.

Although it is an infinite-horizon problem, the seller faces the same problem in each period. Therefore, she only needs to solve a static problem in a stationary setting, and her optimal mechanism sequence is either an infinitely repeated sequence of auctions or an infinitely repeated sequence of posted prices.

Proposition 4. *Suppose $T = \infty$. Let p_∞^* and π_∞^P be the unique solution to*

$$\rho(p_\infty^*) = \delta\pi_\infty^P$$
$$\pi_\infty^P = \pi(P_{p_\infty^*}) + \delta k(p_\infty^*)\pi_\infty^P.$$

Let $c_\infty^* = R(A_{\alpha^{-1}(\delta\pi_\infty^P)}) - \left[1 - \delta k(\alpha^{-1}(\delta\pi_\infty^P))\right]\pi_\infty^P$. For any $c \leq R(A_0)$, let $r_\infty^*(c)$ be the unique solution to

$$\frac{1-\delta}{1-\delta k(r_\infty^*(c))} r_\infty^*(c) = \frac{1 - F(r_\infty^*(c))}{f(r_\infty^*(c))} + \frac{\delta\left[R(A_{r_\infty^*(c)}) - R(P_{r_\infty^*(c)}) - c\right]}{1 - \delta k(r_\infty^*(c))}.$$

A cost c seller's optimal mechanism sequence is an infinitely repeated sequence of $P_{p_\infty^*}$ if $c > c_\infty^*$, of $A_{r_\infty^*(c)}$ if $c < c_\infty^*$, and of $P_{p_\infty^*}$ or $A_{r_\infty^*(c_\infty^*)}$ if $c = c_\infty^*$.

Proof of Proposition 4.. We will first calculate respectively the optimal constant price and the optimal constant reserve price and then compare to see which infinite mechanism sequence generates more expected profit. First, we solve the optimal posted price p^*_∞. Let π^P_∞ denote the expected profit of posting p^*_∞ in each period. Together, p^*_∞ and π^P_∞ must simultaneously satisfy the following equations,

$$\rho(p^*_\infty) = \delta\pi^P_\infty, \tag{11}$$

$$\pi^P_\infty = \pi(P_{p^*_\infty}) + \delta k(p^*_\infty)\pi^P_\infty, \tag{12}$$

where Eq. (11) equates the posted price marginal revenue to the expected profit, and Eq. (12) equates the optimal expected profit to the profit of using $P_{p^*_\infty}$ plus the expected profit if it is not sold. Rearranging Eq. (12) and plugging it into Eq. (11), p^*_∞ is uniquely determined by[10]

$$\frac{1-\delta}{1-\delta[1-F^n(p^*_\infty)]}p^*_\infty - \frac{1-F^n(p^*_\infty)}{[F^n(p^*_\infty)]'} = 0. \tag{13}$$

Similarly, we can calculate the optimal reserve price $r^*_\infty(c)$ for a cost c seller. Let $\pi^A_\infty(c)$ denote the continuation value of running $A_{r^*_\infty(c)}$ each period. Together, $r^*_\infty(c)$ and π^A_∞ must simultaneously satisfy the following two equations,

$$\alpha(r^*_\infty(c)) = \delta\pi^A_\infty(c), \tag{14}$$

$$\pi^A_\infty(c) = \pi(A_{r^*_\infty(c)}) + \delta k(r^*_\infty(c))\pi^A_\infty(c). \tag{15}$$

Substitute Eq. (15) into Eq. (14) and rearrange,

$$\frac{1-\delta}{1-\delta k(r^*_\infty(c))}r^*_\infty(c) = \frac{1-F(r^*_\infty(c))}{f(r^*_\infty(c))} + \frac{\delta\left[R(A_{r^*_\infty(c)}) - R(P_{r^*_\infty(c)}) - c\right]}{1-\delta k(r^*_\infty(c))}. \tag{16}$$

The LHS is increasing and the RHS is decreasing in $r^*_\infty(c)$; the LHS is 0 and the RHS is $1/f(0) + \delta[R(A_0) - c]/(1-\delta)$ when $r^*_\infty(c) = 0$ and the LHS is 1 and the RHS is $-\delta c$ when $r^*_\infty(c) = 1$. Therefore, if $c \le R(A_0)$, $r^*_\infty(c)$ is uniquely determined by the system of equations. If $c > R(A_0)$, the seller chooses a posted price for sure, because for any r, A_r generates strictly less expected profit than P_r, as

$$\pi(A_r) - \pi(P_r) \le \pi(A_0) - \pi(P_0) = R(A_r) - c < 0.$$

Therefore, there is a cutoff cost c^*_∞ such that the seller runs $A_{r^*_\infty(c)}$ if $c > c^*_\infty$, runs $P_{p^*_\infty}$ if $c < c^*_\infty$ and is indifferent between the two if $c = c^*_\infty$. c^*_∞ must satisfy $\pi^A_\infty(c^*_\infty) = \pi^P_\infty$. There is an easy way to calculate c^*_∞ than solving Eqs. (15) and (16). $r^*_\infty(c^*_\infty)$ is determined by Eq. (15),

$$\alpha(r^*_\infty(c^*_\infty)) = \delta\pi^P_\infty, \tag{17}$$

but c^*_∞ must satisfy that in any period, the seller is indifferent between posting the optimal price and running the optimal auction instead, so $\pi^P_\infty = R(A_{r^*_\infty(c^*_\infty)}) - c^*_\infty + \delta k(r^*_\infty(c^*_\infty))\pi^P_\infty$, or

$$c^*_\infty = R(A_{\alpha^{-1}(\delta\pi^P_\infty)}) - \left[1-\delta k(\alpha^{-1}(\delta\pi^P_\infty))\right]\pi^P_\infty. \tag{18}$$

□

We also have the following comparative statics result.

Proposition 5. *Suppose $T = \infty$. When the seller is more patient, the cutoff cost c^*_∞ decreases.*

Proof of Proposition 5.. Let $r^*_\infty \equiv r^*_\infty(c^*_\infty)$ throughout the proof. Differentiate c^*_∞ with respect to δ in Eq. (18),

$$\begin{aligned}\frac{dc^*_\infty}{d\delta} &= -\alpha(r^*_\infty)k'(r^*_\infty) - [1-\delta k(r^*_\infty)]\frac{d\pi^P_\infty}{d\delta} - \left[-k(r^*_\infty) - \delta k'(r^*_\infty)\right]\pi^P_\infty \\ &= k(r^*_\infty)\left[-\alpha(r^*_\infty) + \delta\pi^P_\infty\right] + k(r^*_\infty)\pi^P_\infty - [1-\delta k(r^*_\infty)]\frac{d\pi^P_\infty}{d\delta}.\end{aligned}$$

The first term $k(r^*_\infty)\left[-\alpha(r^*_\infty) + \delta\pi^P_\infty\right] = 0$ by Eq. (17), so

$$\frac{dc^*_\infty}{d\delta} = k(r^*_\infty)\left[\pi^P_\infty + \delta\frac{d\pi^P_\infty}{d\delta}\right] - \frac{d\pi^P_\infty}{d\delta}. \tag{19}$$

By Eqs. (11) and (17), and the facts that $\alpha(\cdot) \ge \rho(\cdot)$, we have $r^*_\infty \le p^*_\infty$ and consequently $k(r^*_\infty) \le k(p^*_\infty)$. Therefore,

$$\frac{dc^*_\infty}{d\delta} \le k(p^*_\infty)\left[\pi^P_\infty + \delta\frac{d\pi^P_\infty}{d\delta}\right] - \frac{d\pi^P_\infty}{d\delta}.$$

[10] The first term of Eq. (13) is strictly increasing in and the second term is strictly decreasing in p^*_∞, and when $p^*_\infty = 0$, the LHS is negative but when $p^*_\infty = 1$ the LHS is positive.

It suffices to show that the RHS is negative.

Rearrange Eq. (12),

$$[1-\delta k(p_\infty^*)]\pi_\infty^P = p_\infty^*[1-k(p_\infty^*)],$$

and differentiate it with respect to δ,

$$\frac{d\pi_\infty^P}{d\delta}[1-\delta k(p_\infty^*)] - \pi_\infty^P\left[k(p_\infty^*)+k'(p_\infty^*)\right] = -p_\infty^* k'(p_\infty^*) + [1-k(p_\infty^*)]\frac{dp_\infty^*}{d\delta}.$$

Rearrange, we obtain

$$k(p_\infty^*)\left[\pi_\infty^P + \delta\frac{d\pi_\infty^P}{d\delta}\right] - \frac{d\pi_\infty^P}{d\delta} = \left[\rho(p_\infty^*) - \pi_\infty^P\right]k'(p_\infty^*)\frac{dp_\infty^*}{d\delta}.$$

The RHS by Eq. (11) equals $(\delta-1)\pi_\infty^P k'(p_\infty^*)\frac{dp_\infty^*}{d\delta}$. Differentiate Eq. (11) with respect to δ,

$$\pi_\infty^P + \delta\frac{d\pi_\infty^P}{d\delta} = \frac{\partial\rho(p_\infty^*)}{\partial p_\infty^*}\frac{dp_\infty^*}{d\delta} > 0.$$

$\rho' > 0$ implies $dp_\infty^*/d\delta > 0$, which implies $(\delta-1)\pi_\infty^P k'(p_\infty^*)\frac{dp_\infty^*}{d\delta} < 0$ as $\delta < 1$. □

The intuition of the result is as follows. Take the extreme case that the seller is infinitely patient ($\delta = 1$). As there is an infinite number of periods, essentially there is an infinite stream of buyers. Then posting a price very close to 1 generates as much revenue as running the optimal auction, which also gives an expected revenue close to 1.

The result sheds light on the declining use of auctions on eBay. As eBay became a more popular platform, more and more sellers switched to simple prices from auctions. In January 2002, more than 90% of the active listings on eBay were auctions, but by late 2012, only 10% of the active listings were auctions, and the rest were simple posted prices and the hybrid Buy It Now formats (Einav et al., 2018). An increase in seller's patience can be thought of as an increase in the transaction speed or a reduction in the transaction period. The result implies that as the market becomes thicker, even without a change in sellers' cost composition, more sellers will choose to post prices. Simply an increase in the size of the market can alter the seller's choice of mechanism.

Appendix D. Long-lived buyers

D1. Long-lived myopic buyers

Proof of Proposition 2. Suppose that $(A_{r_1^*}, P_{p_2^*})$ is optimal.

If $r_1^* < p_2^*$, the first period buyers do not buy in the second period, so the mechanism sequence $(A_{r_1^*}, P_{p_2^*})$ generates the same profit as in the setting where all the sellers are short-lived. As the proof of Corollary 1 has shown, $(A_{r_1^*}, A_{p_2^*})$ cannot simultaneously dominate $(A_{r_1^*}, A_{r_2^*=\alpha^{-1}(0)})$ and $(P_{p_2^*}, P_{p_2^*})$ in short-lived buyers setting. $(A_{r_1^*}, A_{r_2^*})$ generates higher payoff in the current long-lived myopic buyers setting than in the short-lived buyers setting since some buyers born in the first period buy in the second period.

If $r_1^* \geq p_2^*$, then the following contradiction shows that $(A_{r_1^*}, P_{p_2^*})$ cannot be optimal. The optimal mechanism sequence $(A_{r_1^*}, P_{p_2^*})$ must dominate $(A_{r_1^*}, A_{p_2^*})$,

$$\pi(A_{r_1^*}, P_{p_2^*}) - \pi(A_{r_1^*}, A_{p_2^*}) = -F^n(r_1^*)[R_2(P_{r_1^*}, A_{p_2^*}) - R_2(A_{r_1^*}, P_{p_2^*}) - c] \geq 0$$

where $R_2(P_{r_1^*}, A_{p_2^*})$ and $R_2(A_{r_1^*}, P_{p_2^*})$ denote the second-period revenue with the mechanism sequence $(P_{r_1^*}, A_{p_2^*})$ and $(A_{r_1^*}, P_{p_2^*})$. $(A_{r_1^*}, P_{p_2^*})$ must also dominate $(P_{r_1^*}, P_{p_2^*})$,

$$\begin{aligned}
&\pi(A_{r_1^*}, P_{p_2^*}) - \pi(P_{r_1^*}, P_{p_2^*}) \\
&= [R(A_{r_1^*}) - c + F^n(r_1^*)R_2(A_{r_1^*}, P_{p_2^*})] - [R(P_{r_1^*}) + F^n(r_1^*)R_2(P_{r_1^*}, P_{p_2^*})] \\
&= R(A_{r_1^*}) - R(P_{r_1^*}) - c \geq 0
\end{aligned}$$

Therefore, we must have

$$R(A_{r_1^*}) - R(P_{r_1^*}) \geq c \geq R_2(P_{r_1^*}, A_{p_2^*}) - R_2(P_{r_1^*}, P_{p_2^*}). \tag{20}$$

$R(A_{r_1^*}) - R(P_{r_1^*}) \leq R(A_{p_2^*}) - R(P_{p_2^*}) \leq R_2(P_{r_1^*}, A_{p_2^*}) - R_2(P_{r_1^*}, P_{p_2^*})$, a contradiction. The second inequality holds because of the extra buyers in the second period (for any x, the revenue difference between the reserve price x auction and the posted price x increases in the number of buyers).

In summary, the optimal auction-price sequence is shown to be dominated by at least one other mechanism. Therefore, such a sequence can never be optimal.

Let $R_{AA}^*(c)$ denote the revenue of the optimal auction-auction sequence when the auction cost is c. Let $R_{PA}^*(c)$ and $R_{PP}^*(c)$ be similarly defined. We know that $R_{AA}^*(0) > R_{PA}^*(0) > R_{PP}^*(0)$. In addition, $dR_{AA}^*(c)/dc < -1$, $-1 < dR_{PA}^*(c)/dc < 0$, and $R_{PP}^*(c) = 0$. The unambiguous ordering in the decreases in the slopes guarantees that there exist c_1^* and c_2^* such that for cost $c \leq c_1^*$, $R_{AA}^*(c) \geq R_{PA}^*(c), R_{PP}^*(c)$; for cost $c_1^* < c \leq c_2^*$, $R_{PA}^*(c) \geq R_{AA}^*(c), R_{PP}^*(c)$; and for $c > c_2^*$, $R_{PP}^*(c) \geq R_{AA}^*(c), R_{PA}^*(c)$. □

D2. Long-lived forward-looking buyers

Proof of Proposition 3.. Suppose that $(A_{r_1^*}, P_{p_2^*})$ is optimal.

If $r_1^* \leq p_2^*$, the profit generated is the same as in the case when buyers are long-lived and myopic, and also the same as in the case when buyers are short-lived. The proof that $(A_{r_1^*}, P_{p_2^*})$ cannot be optimal follows the proof of Proposition 2.

If $r_1^* > p_2^*$, let $\tilde{v} \geq r_1^*$ denote the value of the buyer who is indifferent between buying and not buying in the first period in the mechanism sequence $(A_{r_1^*}, P_{p_2^*})$. Let $(P_{p_1}, P_{p_2^*})$ be the price-price mechanism sequence where the buyer of value $\tilde{v}$ is indifferent between buying and not buying. The optimality of $(A_{r_1^*}, P_{p_2^*})$ implies that $(A_{r_1^*}, P_{p_2^*})$ dominates $(P_{p_1}, P_{p_2^*})$, i.e.

$$\pi(A_{r_1^*}, P_{p_2^*}) - \pi(P_{p_1}, P_{p_2^*}) = R(A_{\tilde{v}}) - R(P_{\tilde{v}}) - c > 0.$$

Let $(A_{r_1^*}, A_{r_2})$ be the auction-auction mechanism sequence in which the buyer of value $\tilde{v}$ is indifferent between buying and not buying in the first period.

$$\pi(A_{r_1^*}, P_{p_2^*}) - \pi(A_{r_1^*}, A_{r_2}) = -F^n(\tilde{v})[R_2(A_{r_1^*}, A_{r_2}) - R_2(A_{r_1^*}, P_{p_2^*}) - c] > 0$$

where $R_2(A_{r_1^*}, A_{r_2})$ and $R_2(A_{r_1^*}, P_{p_2^*})$ are the revenues. The two inequalities together imply

$$R(A_{\tilde{v}}) - R(P_{\tilde{v}}) > c > R_2(A_{r_1^*}, A_{r_2}) - R_2(A_{r_1^*}, P_{p_2^*}) > 0.$$

Since $(A_{r_1^*}, P_{p_2^*})$ is optimal, $R_2(A_{r_1^*}, P_{p_2^*})$ is bigger than the revenue $\tilde{R}$ generated by P_{r_2} when there are n buyers with values smaller than $\tilde{v}$ and n new born buyers with values drawn from distribution F. $R_2(A_{r_1^*}, A_{r_2}) - R_2(A_{r_1^*}, P_{p_2^*}) > R_2(A_{r_1^*}, A_{r_2}) - \tilde{R}$. Since $\tilde{v} > r_2$, $R_2(A_{r_1^*}, A_{r_2}) - \tilde{R} > R(A_{\tilde{v}}) - R(P_{\tilde{v}})$, contradicting the inequality above. Therefore, an auction-price sequence cannot be optimal.

Let $R^*_{AA}(c)$ denote the revenue of the optimal auction-auction sequence when the auction cost is c. Let $R^*_{PA}(c)$ and $R^*_{PP}(c)$ be similarly defined. We know that $R^*_{AA}(0) > R^*_{PA}(0) > R^*_{PP}(0)$. In addition, $dR^*_{AA}(c)/dc < -1$, $-1 < dR^*_{PA}(c)/dc < 0$, and $R^*_{PP}(c) = 0$. The unambiguous ordering in the decreases in the slopes guarantees that there exist c_1^* and c_2^* such that for cost $c \leq c_1^*$, $R^*_{AA}(c) \geq \max\{R^*_{PA}(c), R^*_{PP}(c)\}$; for cost $c_1^* < c \leq c_2^*$, $R^*_{PA}(c) \geq \max\{R^*_{AA}(c), R^*_{PP}(c)\}$; and for cost $c > c_2^*$, $R^*_{PP}(c) \geq \max\{R^*_{AA}(c), R^*_{PA}(c)\}$. □

References

Anwar, S., Zheng, M., 2015. Posted price selling and online auctions. Games Econ. Behav. 90, 81–92.
Bajari, P., Hortacsu, A., 2004. Economic insights from internet auctions. J. Econ. Lit. 42 (2), 457–486.
Bauner, C., 2015. Mechanism choice and the buy-it-now auction: A structural model of competing buyers and sellers. Int. J. Ind. Organ. 38, 19–31.
Budish, E.B., Takeyama, L.N., 2001. Buy prices in online auctions: irrationality on the internet? Econ. Lett. 72 (3), 325–333. doi:10.1016/S0165-1765(01)00438-4.
Bulow, J., Roberts, J., 1989. The simple economics of optimal auctions. J. Polit. Econ. 97 (5), 1060–1090.
Burguet, R., 1996. Optimal repeated purchases when sellers are learning about costs. J. Econ. Theory 68, 440–455.
Crémer, J., Speigel, Y., Zheng, C.Z., 2009. Auctions with costly information acquisition. Econ. Theory 38, 41–72.
Crémer, J., Spiegel, Y., Zheng, C.Z., 2007. Optimal search auctions. J. Econ. Theory 134, 226–248.
Dilmé, F., Li, F., 2019. Revenue management without commitment: dynamic pricing and periodic flash sales. Rev. Econ. Stud. 86 (5), 1999–2034.
Einav, L., Farronato, C., Levin, J., Sundaresan, N., 2018. Auctions versus posted prices in online markets. J. Polit. Econ. 126 (1), 178–215.
Hammond, R.G., 2013. A structural model of competing sellers: auctions and posted prices. Eur. Econ. Rev. 60, 52–68.
Hart, S., Nisan, N., 2013. The menu-size complexity of auctions. In: EC '13: Proceedings of the Fourteenth ACM Conference on Electronic Commerce, pp. 565–566.
Hu, Y., Zhang, H., 2020. Overcoming borrowing stigma: the design of lending-of-last-resort policies. Mimeo.
Lee, J., Li, D. Z., 2020. Search for bidders by a deadline. Mimeo.
Mathews, T., 2004. The impact of discounting on an auction with a buyout option: a theoretical analysis motivated by ebay's buy-it-now feature. J. Econ. 81 (1), 25–52.
McAfee, P., McMillan, J., 1988. Search mechanisms. J. Econ. Theory 44, 99–123.
Myerson, R., 1981. Optimal auction design. Math. Oper. Res. 6 (1), 58–73.
Reynolds, S.S., Wooders, J., 2009. Auctions with a buy price. Econ. Theory 38, 9–39.
Satterthwaite, M.A., Shneyerov, A., 2007. Dynamic matching, two-sided incomplete information, and participation costs: existence and convergence to perfect competition. Econometrica 75 (1), 155–200.
Satterthwaite, M.A., Shneyerov, A., 2008. Convergence to perfect competition of a dynamic matching and bargaining market with two-sided incomplete information and exogenous exit rate. Games Econ. Behav. 63, 435–467.
Skreta, V., 2006. Sequentially optimal mechanisms. Rev. Econ. Stud. 73 (4), 1085–1111.
Standifird, S.S., Roelofs, M.R., Durham, Y., 2005. The impact of ebay's buy-it-now function on bidder behavior. Int. J. Electron. Commerce 9 (2), 167–176.
Weitzman, M.L., 1979. Optimal search for the best alternative. Econometrica 47 (3), 641–654.
Zhang, H., 2015. Essays in Matching, Auctions, and Evolution. University of Chicago PhD Dissertation.
Zhang, H., 2015. The optimal sequence of prices and auctions (extended abstract). In: The Third Conference on Auctions, Market Mechanisms and Their Applications. ACM doi:10.4108/eai.8-8-2015.2260557.
Zhang, H., 2020. Prices versus auctions in large markets. Economic Theory. Forthcoming