Changes in marital sorting: Theory and evidence from the US

Pierre-André Chiappori* Monica Costa Dias†
Costas Meghir‡ Hanzhe Zhang§

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Abstract

Positive assortative matching refers to the tendency of individuals with similar characteristics to form partnerships. Measuring the extent to which assortative matching differs between two economies is challenging when the marginal distributions of the characteristic along which sorting takes place (e.g., education) change for either or both sexes. We show how the use of different measures can generate different conclusions. We provide axiomatic characterization for measures such as the odds ratio, normalized trace, and likelihood ratio, and provide a structural economic interpretation of the odds ratio. We then use our approach to consider how marital sorting by education changed between the 1950s and the 1970s cohort, for which both educational attainment and returns in the labor market changed substantially.

Keywords: sorting, homogamy, educational attainment, marriage markets

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*Columbia University, pc2167@columbia.edu

†University of Bristol and Institute for Fiscal Studies, monica.costa-dias@bristol.ac.uk

‡Yale University, NBER, IFS, IZA, CEPR, IFAU, c.meghir@yale.edu

§Michigan State University, hanzhe@msu.edu

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1 Introduction

The study of sorting in the marriage market has recently attracted renewed attention. The degree of homogamy in marriage—defined as people's tendency to "marry their own"—has important consequences for family inequalities and intergenerational transmission of human capital. Therefore, it is surprising that various studies on this topic have not reached a consensus on the evolution of homogamy in recent years, and even on how best to measure homogamy (e.g., Fernández and Rogerson, 2001; Mare and Schwartz, 2005; Greenwood et al., 2014; Siow, 2015; Chiappori, Salanié, and Weiss, 2017; Eika, Mogstad, and Zafar, 2019; Ciscato and Weber, 2020; Gihleb and Lang, 2020; Hou et al., 2022).

The goal of this paper is to understand why different approaches to the same problem, using similar or even the same data, can reach opposite conclusions and, more generally, to clarify the theoretical issues underlying the choice of a particular measure of assortativeness. Our analysis considers two populations, men and women, sorting in marriage according to a characteristic, say education. Whenever the marginal distributions change—e.g., women's average education increases—matching patterns will change. The main problem faced by any measure of assortativeness is to disentangle the mechanical effects of such variations in the marginal distributions from deeper changes in the matching structure itself—for instance, originating from changes in the gain generated by assortativeness along that characteristic. The latter represents what one would call "changes in the assortativeness." Existing studies propose various measures—indices and rankings—to capture the assortativeness and the changes therein. They measure assortativeness in different ways and may therefore generate divergent conclusions.

Rather than starting with specific measures and justifying them with selected desirable properties they satisfy, we take an *axiomatic approach*: We start with a set of properties that the measures should satisfy and establish how appropriate the measures are for capturing changes in assortative matching through the lenses of these properties.

We consider three sets of basic axioms: (i) axioms of *invariance* to specify when two matching patterns are equally assortative, (ii) axioms of *monotonicity* to specify when one matching pattern is more assortative than another, and (iii) axioms of *homogamy* to specify matching patterns that are most assortative.[1] Namely, invariance axioms are *scale invariance* (invariance to scaling the market), *side invariance* (invariance to swapping sides), and *type invariance* (invariance to swapping types). The two notions of monotonicity we consider are (i) *marginal monotonicity*, ranking two sets of matches of the same marginal distributions, and (ii) *diagonal* and *off-diagonal monotonicity* (assortativeness increases when more like types match and decreases when more unlike types match). Finally, we introduce two definitions of the most assortative matching: *full homogamy* (every pair in the market is homogamous) and *maximum homogamy* (the maximum number of homogamous pairs is achieved in the market, although not every pair is necessarily homogamous).

While some commonly used measures, such as the likelihood ratio, fail to satisfy one or more of the basic axioms, the *odds ratio* (the ratio of college and noncollege individuals' odds to match with the same type and with a different type), the *normalized trace* (the proportion of like types), and the *minimum distance* (the combination of sorting patterns under random matching and maximum positive assortative matching that best fits the empirical observation) satisfy all basic axioms. Since these are genuinely different measures rather than monotonic transformations of each other, they may point in opposite directions when used to investigate changes in assortativeness between two matching patterns. Their differences are empirically relevant. We show estimates of changing marital sorting patterns in the US where one finds instances of these measures reaching opposite conclusions. To distinguish some of these measures, we additionally define a set of *characterization axioms*.

In two-type markets, the complete ranking induced by the odds ratio is the unique one that satisfies the basic axioms plus *marginal independence* (Edwards, 1963) (Theorem 1). Under marginal independence, only the odds of marrying different types of spouses matter for

[1] In this paper, we use axioms and properties interchangeably.

the measure of assortativeness, not the marginal distributions of the types. All indices that are monotonic transformations of the odds ratio (e.g., Yule's Q, Yule's Y, and log odds ratio) provide the same assortativeness ordering. We also provide a structural economic interpretation of the odds ratio: It estimates the average gain per couple from assortative matching compared to nonassortative matching within the Choo and Siow (2006) transferable-utilities matching framework.

We also define *decomposability* axioms: The assortativeness measure for any matching in a market is a weighted average or weighted sum of the measures for the matching patterns in the submarkets decomposed from the market. The *normalized trace*—the proportion of pairs of like types—is the unique index, up to positive affine transformation, that satisfies the basic axioms plus *population decomposability*, i.e., the weight to average is population size (Theorem 2). For comparison, we also provide an axiomatic characterization of the *aggregate likelihood ratio* (used, for example, in Eika, Mogstad, and Zafar, 2019)—the excess likelihood of like types pairing up relative to the hypothetical likelihood under random matching. It is the unique measure, up to positive multiplication, that satisfies selected basic axioms plus *random decomposability*, i.e., the weight to sum is the hypothetical proportion of pairs of like types under random matching (Theorem 3).

For markets with more than two types, the indices such as normalized trace and likelihood ratio can be extended to provide measures of assortativeness.[2] In practice, there are 14 detailed education categories in the US Census, so *robustness to categorization* is an additional desired property: Comparisons of assortativeness should be robust to how education is categorized into two or several groups. We provide an impossibility result: No total preorder satisfies marginal monotonicity (the weaker monotonicity axiom) and robustness to categorization (Theorem 4). As a result, we must resort to partial rankings to evaluate assortativeness of multi-type matching patterns.

The continued attention in the literature to how assortativeness on education has evolved

[2] They can also be modified to compare markets with singles and one-sided (e.g., homosexual) matches (Zhang, 2024).

in the recent past has been fueled by contradictory findings. We revisit this issue using our axiomatic approach, studying changes in assortativeness on education in the US, across cohorts born in the 1950s and 1970s. We contribute to that discussion by arguing that ambiguity can sometimes be resolved by selecting indices that have desirable properties, and by studying local (submarket) changes in assortativeness rather than global ones, which are hard to interpret. The indices that satisfy our basic axioms consistently show that assortativeness by education has increased at the top of the education distribution, but has decreased at the very top, between college graduates and post graduates. We also find that the type-specific and aggregate likelihood ratios—which fail to satisfy a number of the basic axioms of invariance, monotonicity, and homogamy—frequently contradict the findings of other indices and of each other.

The rest of the paper is organized as follows. Section 2 sets up the model and discusses special matching patterns and commonly used measures. Section 3 lays out the basic axioms. Section 4 discusses the axiomatic characterization results for the odds ratio, normalized trace, and aggregate likelihood ratio. Section 5 discusses the structural interpretation of the odds ratio. Section 6 discusses the theoretical results with more than two types. Section 7 provides empirical results on marital sorting in the US, and Section 8 concludes.

2 Model, matchings, and measures

2.1 Model and objective

Each man and woman possesses a trait (e.g., college education or any other observable psychological, biological, or socioeconomic trait). We start with the setting where a trait is one of two types, e.g., θ_1 and θ_2. For expositional ease and alignment with our primary empirical application, we refer to the two traits as college-educated and noncollege-educated.

Consider the matching between men and women described by matrix $M = (a, b, c, d)$:

$$M = \begin{array}{ccc} m\backslash w & \theta_1 \text{ (college)} & \theta_2 \text{ (noncollege)} \\ \theta_1 \text{ (college)} & a & b \\ \theta_2 \text{ (noncollege)} & c & d \end{array} .$$

A cell describes the mass of pairs between a specific combination of types of men and women. To recover the marginal distribution, i.e., mass of (matched) individuals of a specific gender and type, we sum a column or a row. For example, there is mass $a+b$ of college men. The marginal distribution for men is then described by $(a+b, c+d)$ and that for women is $(a+c, b+d)$. Assume a full support of types on both sides of the market: $(a+b)(a+c)(b+d)(c+d) > 0$. Let $|M| \equiv a+b+c+d$ denote its population size.

We define *positive matching matrices* $M = (a, b, c, d)$, i.e., matrices such that each of the entries a, b, c, and d is strictly positive.[3] We denote them by $M > \mathbf{0}$, where bold $\mathbf{0}$ denotes the zero matrix, the 2×2 zero matrix in this context.

Objective. Our aim is to define a set of axioms that a measure of assortative matching should reasonably satisfy and to use these to distinguish among different measures, which often lead to contradictory results. A measure may be an indexing, a complete ranking, or a partial ranking of all matching patterns.

Formally, let $\mathcal{M} = \mathbb{R}^4_+ \backslash \{\mathbf{0}\}$ denote the entire collection of possible matching patterns in two-type markets. A measure of assortativeness may be (i) an *index* $I : \mathcal{M} \to \mathbb{R}_+$; (ii) a *total preorder* (i.e., a complete ranking) on $\mathcal{M}$, a binary relation that satisfies totality ($M \succeq M'$ or $M' \succeq M$) and transitivity ($M \succeq M'$ and $M' \succeq M''$ imply $M \succeq M''$); or (iii) a *preorder* (i.e., a partial or complete ranking) on $\mathcal{M}$, a binary relation that satisfies reflexivity ($M \succeq M$) and transitivity.[4] Any index is associated with a total preorder, so studying the ordinal properties of an index is akin to studying the properties of the total preorder it induces.

[3]Note that positive matching matrices are distinct from positive definite matrices.

[4]We consider preorders rather than orders (which satisfy the additional condition of antisymmetry: $M \succeq M'$ and $M' \succeq M$ imply $M = M'$) because there may be equally assortative matching patterns.

2.2 Special matching patterns

Fully and maximally assortative matching. When a person is matched with a partner of the same type with probability one, i.e., $b = c = 0$, we call such a matching *fully positive-assortative*. However, fully positive-assortative matching is feasible only when men and women have the same distribution of education. More generally, we call a matching *maximally positive-assortative* if there is a maximum mass of pairs of like types, holding marginal distributions fixed: $bc = 0$ (that is, $b = 0$ or $c = 0$ or both). Say, all individuals on the short side of the educated category (men in our case) marry an educated partner (i.e., $b = 0$), but not every educated individual on the long side (women) does so (i.e., $c > 0$); intuitively, the only reason that we observe "mixed" couples is the lack of educated men. Every market has a unique maximally positive-assortative matching.

In the two-by-two case, we can analogously define *fully negative-assortative* matching as one in which all individuals are matched with a partner of a different type with probability one, i.e., $a = d = 0$; and *maximally negative-assortative* (or equivalently, *minimally positive-assortative*) matching as one in which there is a maximal mass of pairs of unlike types (or equivalently, in two-by-two markets, minimal mass of pairs of like types), i.e., $ad = 0$.

Note that fully (positive or negative) assortative matching is maximally assortative, but maximally assortative matching is not necessarily fully assortative. Most—not all—of the preorders in use have maximally positive-assortative matching matrices as maximum elements of $\mathcal{M}$.

Positive and negative assortative matching. One way of defining positive and negative assortative matching is to compare to the random matching benchmark. We say that matching M is a *positive assortative matching (PAM)* if the mass of couples with equal education (the "diagonal" of matrix M) is strictly larger than what would obtain under random matching. Under random matching, the mass of couples in which both spouses are college-educated is $|M| \cdot \left[(a+b)(a+c)/|M|^2\right]$. Then we have PAM if and only if

$$a(a+b+c+d) > (a+b)(a+c), \tag{1}$$

or equivalently,

$$ad > bc. \tag{2}$$

This inequality also implies that more noncollege individuals marry each other than would be predicted by random matching.[5] In other words, PAM arises when extra forces generate more matches between equally educated people than would happen for random reasons. *Negative assortative matching (NAM)* in two-type markets can be analogously defined: $ad < bc$.

2.3 Existing measures and their differences

2.3.1 Existing measures

We now review some of the most commonly used measures in the literature. Many are measuring the extent to which the left-hand side is larger than the right-hand side in equations (1) and (2).

Odds ratios. The *odds ratio* is probably the most widely used index:[6]

$$I_O(M) = \begin{cases} \dfrac{ad}{bc} & \text{if } b \neq 0 \text{ and } c \neq 0, \\ +\infty & \text{if } bc = 0. \end{cases}$$

The index ranges from 0 to $+\infty$. A few monotonic transformations of the odds ratio are used in the literature. They all yield the same preorder of assortativeness. *Log odds ratio* is

[5] Mathematically, $d > (d+b)(d+c)/|M| \Leftrightarrow d(a+b+c+d) > (d+b)(d+c) \Leftrightarrow d(a+b) > b(d+c) \Leftrightarrow ad > bc$. The inequality $ad > bc$ also implies $a+d > [(a+b)(a+c)+(d+b)(d+c)]/(a+b+c+d)$.

[6] The odds ratio is popular in the demographic literature, as it can be directly derived from the log-linear approach; see, for instance, Mare (2001), Mare and Schwartz (2005), and Bouchet-Valat (2014). In economics, it was used by Siow (2015) ("local odds ratio"), Chiappori, Salanié, and Weiss (2017), Chiappori et al. (2020), and Ciscato and Weber (2020), among many others.

$I_o(M) \equiv \ln I_O(M)$. *Yule's Q* or *coefficient of association* (Yule, 1900) is $I_Q(M) \equiv \frac{I_O(M)-1}{I_O(M)+1}$. This is +1 when PAM, −1 when NAM, and 0 when random matching (uncorrelated). *Yule's Y* or *coefficient of colligation* (Yule, 1912) is $I_Y(M) \equiv \frac{\sqrt{I_O(M)}-1}{\sqrt{I_O(M)}+1}$.

Likelihood ratios. *Type-specific likelihood ratio*, used by Eika, Mogstad, and Zafar (2019), for instance, measures marital sorting between men and women of the same type by the ratio of the actual probability of matching relative to what would occur at random:[7]

$$I_{L1}(M) \equiv \frac{\text{observed matches } \theta_1\theta_1}{\text{random matching baseline}} = \frac{a}{|M|} \Big/ \frac{a+b}{|M|}\frac{a+c}{|M|} = \frac{a(a+b+c+d)}{(a+b)(a+c)}.$$

The likelihood ratio for population type θ_1 compares the realized mass of college-college pairs with the benchmark of the hypothetical mass of such pairs under random matching. The likelihood ratio for population type θ_2 is analogously defined:

$$I_{L2}(M) \equiv \frac{\text{observed matches } \theta_2\theta_2}{\text{random matching baseline}} = \frac{d}{|M|} \Big/ \frac{d+b}{|M|}\frac{d+c}{|M|} = \frac{d(a+b+c+d)}{(d+b)(d+c)}.$$

One issue with the type-dependent likelihood ratio is that it requires the choice of a type as a benchmark; in other words, it fails *type invariance*, which we will define later on. Our empirical results show that the likelihood ratios based on different types yield opposite conclusions on the direction of change in educational homogamy in the US.

The *aggregate likelihood ratio* is a weighted average of the type-specific likelihood ratios, in which the weight on each type-specific likelihood ratio is the expected mass of pairs of

[7] Ciscato, Galichon, and Goussé (2020) call this the homogamy rate. It is used to study sorting on career ambition (Almar et al., 2023) and on childhood family income percentile (Binder et al., 2023).

like types under random matching.

$$I_L(M) \equiv \frac{(a+b)(a+c)}{(a+b)(a+c)+(d+b)(d+c)} I_{L1}(M) + \frac{(d+b)(d+c)}{(a+b)(a+c)+(d+b)(d+c)} I_{L2}(M)$$

$$= \frac{\text{observed matches } \theta_1\theta_1 \text{ and } \theta_2\theta_2}{\text{random matching baseline}} = \frac{a+d}{|M|} \Big/ \left(\frac{a+b}{|M|}\frac{a+c}{|M|} + \frac{d+b}{|M|}\frac{d+c}{|M|} \right).$$

Simplified, this is the ratio between the observed mass of couples of all like types and the counterfactual mass of pairs if individuals were to match randomly.

Normalized trace. A simple index that turns out to have nice properties is what we call the *normalized trace*.[8] It equals 1 if the matching is maximum PAM, equals 0 if the matching is minimum PAM, and equals the proportion of like types in the market otherwise:

$$I_{tr}(M) = \begin{cases} 1 & \text{if } bc = 0, \\ \frac{a+d}{a+b+c+d} \in (0,1) & \text{if } abcd \neq 0, \\ 0 & \text{if } ad = 0. \end{cases}$$

Minimum distance. In the *minimum distance* approach of Fernández and Rogerson (2001), Abbott et al. (2019), and Wu and Zhang (2021), one constructs the convex combination of two extreme cases—random matching and maximum PAM—that minimizes the distance with the matching under consideration and defines the weight of the maximum PAM component as the index. It coincides with the *perfect-random normalization* of Liu and Lu (2006) and Shen (2020) in two-type markets and is equal to

$$I_{MD}(M) = \frac{ad - bc}{(a + \min\{b,c\})(d + \min\{b,c\})}.$$

[8] Using the normalized trace, Cheremukhin, Restrepo-Echavarria, and Tutino (2024) study marital sorting in the US, and Li (2024) and Li and Derdenger (2024) study matching between students and colleges.

Correlation. Another natural index is the correlation between wife's and husband's education, each considered as a Bernoulli random variable taking the value θ_1 with probability $\frac{a+b}{|M|}$ (resp., $\frac{a+c}{|M|}$) and θ_2 with probability $\frac{c+d}{|M|}$ (resp., $\frac{b+d}{|M|}$):

$$I_{Corr}(M) = \frac{ad - bc}{\sqrt{(a+b)(c+d)(a+c)(b+d)}}.$$

This has been used in various contributions (e.g., Greenwood, Guner, and Knowles, 2003; Greenwood et al., 2014), either explicitly or through a linear regression framework. In our 2×2 case, the correlation index also coincides with Spearman's rank correlation, which exploits the natural ranking of education levels (i.e., college > noncollege). Equivalently, one can consider $\chi^2(M) = [I_{Corr}(M)]^2$.

2.3.2 Measuring changes in sorting with alternative measures: An example

As we show in the empirical section, the various measures can imply opposite results with respect to changes in sorting, an issue that has motivated this paper. We provide a graphical illustration of this difference by comparing the odds ratio and the aggregate likelihood ratio, before we analyze the various measures in relation to a set of axioms.

For the purposes of our example, fix a reference matching, $M = (0.2, 0.1, 0.1, 0.6)$ and restrict attention to matching matrices M' in which $|M'| = 1$ and $b = c = (1 - a - d)/2$. The red curve in Figure 1 illustrates what we call the *iso-assortative curve* of M— the collection of matrices that are equally assortative as M—under the odds ratio. Similarly, the white curve is the iso-assortative curve of M under the aggregate likelihood ratio.

Thus, the orange (yellow) area is the set of matching matrices where sorting is more positive (less positive) assortative than M according to *both* measures. The purple areas are the set of matrices classified as less assortative than M by the aggregate likelihood ratio, but more assortative according to the odds ratio. Notably, the fully assortative matching matrices—$a = 1$ and $d = 0$ or $a = 0$ and $d = 1$—and the matching matrices close to them

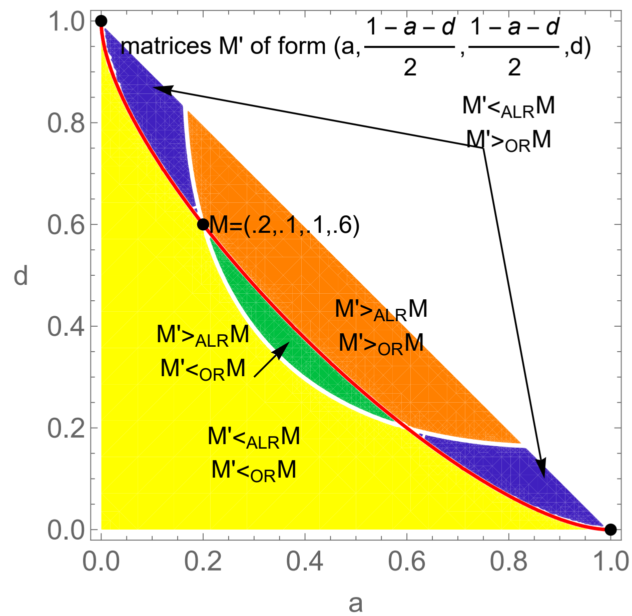

Figure 1: Iso-assortative curves and conflicting region

are deemed strictly less assortative than M under the aggregate likelihood ratio. Finally, the matrices in the green regions are more assortative than M under the aggregate likelihood ratio, and less so under the odds ratio. These differences reflect the way that each of the measures accounts for changes in the marginal distributions.[9]

3 Axioms of invariance, monotonicity, and homogamy

We introduce three sets of basic axioms/properties a measure should satisfy. They describe (i) matching matrices that should be classified as equally assortative, (ii) matching matrices where one must be classified as more assortative than the other, and (iii) matching matrices that should be classified as most assortative. For generality, whenever possible, we define axioms that a preorder should satisfy, rather than an index. We say that a total preorder $\succeq$ is *induced by* index I if for all matching matrices M and M', $I(M) > I(M') \Leftrightarrow M \succ M'$

[9]In general, the iso-assortative curves of a fixed matching matrix can be derived for any measure and its upper contour set can be used to compare and contrast any pairs of measures. In addition, when we relax the restriction of $b = c$, we can construct iso-assortative surfaces and upper contour sets of M in three-dimensional planes. We illustrate the conflicts between aggregate likelihood ratio and odds ratio in the three-dimensional space in Appendix Figure 2.

and $I(M) = I(M') \Leftrightarrow M \sim M'$. Two indices I and I' are *preorder-equivalent* if the total preorders $\succeq_I$ and $\succeq_{I'}$ they induce are equivalent. An index is said to satisfy an axiom if the total preorder induced by the index satisfies it.

3.1 Invariance axioms

We start with axioms that define two equally assortative matching matrices. First, when a matching matrix is scaled up or down by a positive constant without changing the relative composition, the assortativeness should be evaluated as unchanged.

Scale Invariance. *For all M and $\lambda > 0$, $M \sim \lambda \cdot M$.*

Given scale invariance, we can transform M into a contingency table by dividing each cell by $|M|$. Next, the assortativeness should not be affected if we change the sides of the market. Effectively, we switch the masses of high-low (b) and low-high (c) pairs.

Side Invariance. *The matching stays equally assortative when sides are switched:*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \sim \begin{array}{c|c} a & c \\ \hline b & d \end{array}.$$

Neither should switching the types affect the assortativeness. Swapping the labels of the two types is essentially swapping high-high (a) and low-low (d) pairs and swapping high-low (b) and low-high (c) pairs.

Type Invariance. *The matching stays equally assortative when types are switched:*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \sim \begin{array}{c|c} d & c \\ \hline b & a \end{array}.$$

We refer to all three together as *invariance axioms*. These are basic axioms an appropriate measure should satisfy. While type-specific likelihood ratio fails type invariance, all aforementioned measures satisfy all three axioms of invariance (see summary in Table 1).

3.2 Monotonicity axioms

Next, we define notions of monotonicity that specify when one matching matrix is more assortative than another. We define two notions of monotonicity, one based on comparisons of matching matrices with the same marginal distributions and the other based on comparisons of matching matrices with different marginals.

We compare positive matching matrices (all entries are positive). They are the non-maximally assortative matching matrices.

First, consider one notion of monotonicity that compares two matching matrices of the same marginal distributions. Matching matrices that share the same marginal distributions as (a, b, c, d) are of the form $(a + \epsilon, b - \epsilon, c - \epsilon, d + \epsilon)$—essentially, a one-parameter family.

Marginal Monotonicity. *For any positive matching matrices* $M = (a, b, c, d) > \mathbf{0}$ *and* $M' = (a', b', c', d') > \mathbf{0}$ *with the same marginal distributions (i.e.,* $a + c = a' + c'$, $a + b = a' + b'$, $d + b = d' + b'$, *and* $d + c = d' + c'$*),* $M \succ M'$ *if and only if* $a > a'$ *(equivalently,* $b < b'$, $c < c'$, *or* $d > d'$*).*

All aforementioned measures satisfy marginal monotonicity.

Next, we compare matching matrices that differ in marginal distributions, but differ in a minimal way: by one cell. For matching matrices that are not maximally assortative, when pairs of like types (i.e., the terms on the diagonal of the matrix) increase, the matching becomes more assortative.

Diagonal Monotonicity. *For any positive matching matrix $M > \mathbf{0}$ and $\epsilon > 0$,*

$$\begin{array}{c|c} a+\epsilon & b \\ \hline c & d \end{array} \succ \begin{array}{c|c} a & b \\ \hline c & d \end{array} \text{ and } \begin{array}{c|c} a & b \\ \hline c & d+\epsilon \end{array} \succ \begin{array}{c|c} a & b \\ \hline c & d \end{array}.$$

We analogously define *off-diagonal monotonicity*: When pairs of unlike types (i.e., the terms off the diagonal of the matrix) increase, the matching becomes less assortative.

Off-Diagonal Monotonicity. *For any positive matching matrix $M > \mathbf{0}$ and $\epsilon > 0$,*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \succ \begin{array}{c|c} a & b+\epsilon \\ \hline c & d \end{array} \text{ and } \begin{array}{c|c} a & b \\ \hline c & d \end{array} \succ \begin{array}{c|c} a & b \\ \hline c+\epsilon & d \end{array}.$$

Claim 1. *Diagonal and off-diagonal monotonicities imply marginal monotonicity.*

Proof of Claim 1. Suppose positive matrices $M = (a, b, c, d)$ and $M' = (a', b', c', d')$ have the same marginal distributions, and suppose $a > a'$. Because they share the same marginals, $a - a' = b' - b = c' - c = d - d'$ (derived from $a + c = a' + c'$, $a + b = a' + b'$, $d + b = d' + b'$, and $d + c = d' + c'$). Hence, M' can be represented as

$$M' = \begin{array}{c|c} a-(a-a') & b+(b'-b) \\ \hline c+(c'-c) & d-(d-d') \end{array} = \begin{array}{c|c} a-(a-a') & b+(a-a') \\ \hline c+(a-a') & d-(a-a') \end{array}.$$

Then,

$$M' = \begin{array}{c|c} a-(a-a') & b+(a-a') \\ \hline c+(a-a') & d-(a-a') \end{array} \prec \begin{array}{c|c} a & b+(a-a') \\ \hline c+(a-a') & d \end{array} \prec \begin{array}{c|c} a & b \\ \hline c & d \end{array} = M,$$

where the first $\prec$ follows from diagonal monotonicity and the second $\prec$ follows from off-diagonal monotonicity. □

Conversely, however, marginal monotonicity does not necessarily imply diagonal or off-diagonal monotonicity, because marginal monotonicity solely specifies relations for matchings with the same marginal distributions and by itself has no implications for markets with different marginal distributions.

Let us examine whether the aforementioned measures satisfy these monotonicity axioms. The type-specific likelihood ratio satisfies off-diagonal monotonicity but fails diagonal monotonicity: $I_{L1}(1,1,1,6) = 2.25 > I_{L1}(2,1,1,6) \approx 2.22 > I_{L1}(6,1,1,6) \approx 1.74$. To investigate further, we can take the partial derivative of I_{L1} with respect to a. Diagonal monotonicity tends to fail when one of the cells is relatively small. The aggregate likelihood ratio satisfies off-diagonal monotonicity but fails diagonal monotonicity:

$$\frac{\partial I_L}{\partial a} = \frac{(a+d)(d-a)(b+c)}{[(a+b)(a+c)+(d+b)(d+c)]^2} < 0 \text{ when } a > d.$$

The other aforementioned measures satisfy diagonal and off-diagonal monotonicity.

3.3 Homogamy axioms

A natural requirement is that any maximally positive-assortative matching should be a maximal element for preorder $\succeq$. Following Robbins (2009), we call it the *maximum homogamy* property[10]

Maximum Homogamy. *For any M, $ad > 0$, and $bc = 0$: $(a,b,c,d) \succeq M$.*

A less restrictive condition suggests that fully positive-assortative matching should be perceived as the most positive-assortative; we call this the *full homogamy* property.[11]

Full Homogamy. *For any M and $ad > 0$: $(a,0,0,d) \succeq M$.*

[10]HMax in Chiappori, Costa Dias, and Meghir (2022).
[11]Weak HMax in Chiappori, Costa Dias, and Meghir (2022).

Note that maximum homogamy is a stricter condition than full homogamy, because it requires not only matrices of the form $(a, 0, 0, d)$ to be maximal elements, but also matrices of the form $(a, 0, c, d)$ or $(a, b, 0, d)$.

One can readily check that the odds ratio satisfies maximum homogamy, and hence full homogamy, since it becomes infinite when $b = 0$ or $c = 0$. By construction, we have $I_{MD}(M) \leq 1$ for all M and $I_{MD}(a, b, 0, d) = 1$. Therefore minimum distance satisfies maximum homogamy. The correlation index obviously satisfies full homogamy, since the correlation is then equal to 1. However, it does not satisfy maximum homogamy. For instance, $I_{Corr}(45, 5, 5, 45) = .8 > .41 = I_{Corr}(40, 40, 0, 20)$.

Likelihood ratios fail full homogamy, and hence maximum homogamy. Compare matching matrices $M = (5, 5, 5, 85)$ and $M' = (50, 0, 0, 50)$ corresponding, for instance, to two different cohorts in the same economy. The distribution of education is independent of gender in both M and M', but the proportion of educated people has increased from 10% in M to 50% in M'. Cohort M exhibits PAM in the usual sense (more people on the diagonal than would obtain under random matching); yet, 50% of educated people marry an uneducated spouse. Cohort M' displays perfect sorting, with all educated individuals marrying together. The college-based likelihood ratio yields $I_{L1}(M) = 5$ and $I_{L1}(M') = 2$: Assortativeness has *decreased* from M to M'.

3.4 Summary

Table 1 summarizes whether the measures satisfy the basic invariance, monotonicity, and homogamy axioms. The type-specific likelihood ratio does not satisfy type invariance, because relabeling the types would change the measure. The aggregate likelihood ratio fails diagonal monotonicity, maximum homogamy, and full homogamy. The correlation index fails maximum homogamy.

The odds ratio, normalized trace, and minimum distance satisfy all the basic axioms. That begs the question, what would be axioms that can uniquely characterize these mea-

Table 1: Do the measures satisfy the invariance, monotonicity, and homogamy axioms?

	Invariance axioms			Monotonicity axioms			Homogamy axioms	
	Scale	Side	Type	Marginal	Diagonal	Off-diagonal	Maximum	Full
Odds ratio	✓	✓	✓	✓	✓	✓	✓	✓
Type-specific likelihood ratio	✓	✓	X	✓	X	✓	X	X
Aggregate likelihood ratio	✓	✓	✓	✓	X	✓	X	X
Normalized trace	✓	✓	✓	✓	✓	✓	✓	✓
Minimum distance	✓	✓	✓	✓	✓	✓	✓	✓
Correlation	✓	✓	✓	✓	✓	✓	X	✓

sures? For the odds ratio and the normalized trace, we answer this question next.

4 Characterization results

Table 1 shows that, except for the type-specific likelihood ratio, all the measures listed satisfy the invariance axioms and at least one notion of monotonicity: marginal monotonicity. Hence, the basic axioms are not sufficient to distinguish the measures and provide a definitive answer regarding what measure to use.

We need additional axioms to distinguish and characterize the different measures. We will introduce what we call *characterization axioms*, such that a measure will be the unique one that satisfies both the basic axioms and an additional axiom, which essentially highlights the special property of the measure.

4.1 Marginal independence and odds ratio

We first provide the characterization result regarding marginal independence (Edwards, 1963) and the odds ratio. A measure satisfies marginal independence if multiplying any row or any column of any matching matrix does not change the assortativeness order. Intuitively, two matching matrices differing in the number of (say) educated women will be equally assortative if educated women match to educated and uneducated men in the same proportion in both markets. This will happen, for instance, if there is an increase in the number of educated women in the market, and the sorting patterns of the additional educated women are the same as for the initial stock.

Marginal Independence (Edwards, 1963). *For any M and $\lambda > 0$,*

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \sim \begin{array}{c|c} \lambda a & \lambda b \\ \hline c & d \end{array} \sim \begin{array}{c|c} a & b \\ \hline \lambda c & \lambda d \end{array} \sim \begin{array}{c|c} \lambda a & b \\ \hline \lambda c & d \end{array} \sim \begin{array}{c|c} a & \lambda b \\ \hline c & \lambda d \end{array}.$$

Note that marginal independence is an ordinal property. Nonetheless, it is a strong condition. It implies scale, side, and type invariance. In addition,

Claim 2. *Marginal independence and marginal monotonicity imply diagonal monotonicity and off-diagonal monotonicity.*

In fact, more strongly, the three monotonicity properties are equivalent under marginal independence:

Claim 3. *Suppose a measure satisfies marginal independence. It satisfies diagonal monotonicity if and only if it satisfies off-diagonal monotonicity if and only if it satisfies marginal monotonicity.*

Theorem 1. *The odds ratio induces the unique preorder that satisfies marginal monotonicity, marginal independence, and maximum homogamy.*

Note that the sole role of maximum homogamy is to specify the maximum PAM to be the maximal element of the preorder.

It is interesting to refer to an older statistics literature that discusses the properties of measures of association in the case of paired attributes (i.e., in our case, husband's and wife's education). The property posed by Edwards (1963) states that the association should not be "*influenced by the relative sizes of the marginal totals*" (p. 110). That is, the measure should not change if one starts from a matching M and doubles the mass of couples in which the man is educated (while keeping unchanged the ratio of educated versus uneducated wives of educated men).

Edwards (1963) justifies this property by posing that the measure must only be a function of *the proportion of educated women whose husband is educated and the proportion of uneducated women whose husband is educated* (and conversely), so that any population change that keeps these proportions constant should not affect the index. This imposes, in particular, that global changes in the marginal distributions of the characteristic on which people match, for instance a global increase in the mass of educated women, should not systematically impact the index; only changes in the odds of marrying different types of spouses should matter. The condition was later generalized by Altham (1970) to the $n \times n$ case.

Among the indices just reviewed, only the odds ratio and its preorder-preserving transformations satisfy marginal independence. It is interesting to consider how the other indices violate this requirement. Consider matching matrices $M_\lambda = (\lambda a, \lambda b, c, d)$ with $ad > bc$ (PAM) and $\lambda \geq 1$. Suppose λ increases. The type-θ_1 likelihood ratio *decreases* since $\partial I_{L1}/\partial \lambda < 0$, while the type-$\theta_2$ likelihood ratio *increases*. All other indices may increase or decrease, depending on parameters.

4.2 Population decomposability and normalized trace

The next set of characterization axioms will endow the measures with cardinal interpretations. Namely, the assortativeness of a matching matrix will be a weighted average or sum of the measures of matching matrices in the submarkets decomposed from the original market. Depending on the weights used, different decomposability axioms correspond to different measures.

Consider the following axiom in which the weight to average is the population size of submarkets.

Population Decomposability. For any positive matching $M > \mathbf{0}$ and $M' > \mathbf{0}$,

$$I(M + M') = \frac{|M|}{|M|+|M'|} I(M) + \frac{|M'|}{|M|+|M'|} I(M'). \tag{3}$$

Equivalently, for any positive matchings $M_n > \mathbf{0}$, $n = 1, 2, \ldots, N$,

$$I\left(\sum_n M_n\right) = \sum_n \left[\frac{|M_n|}{\sum_n |M_n|} I(M_n)\right]. \tag{4}$$

Equivalent expressions (3) and (4) mean that for any decomposition of a market, the assortativeness index of the matching in the entire market is a weighted average of the same index in its decomposed submarkets.[12] Practical and relevant examples of the decomposition of marriage markets include (i) by geographic regions, (ii) by metropolitan status, such as urban and rural marriages, or (iii) by race, such as into marriages between whites, those between nonwhites, and interracial marriages between whites and nonwhites.

As explained above, marginal monotonicity and the invariance axioms cannot imply diagonal monotonicity or off-diagonal monotonicity. However,

Claim 4. *Marginal monotonicity and population decomposability, together with scale invariance and type invariance, imply diagonal monotonicity and off-diagonal monotonicity.*

We show in the theorem below that population decomposability is a characterization axiom for normalized trace. That is, the normalized trace is the unique index, up to positive

[12] Expressions (3) and (4) are equivalent for the following reasons. It is straightforward that expression (4) implies expression (3), because expression (4) when $n = 2$ is expression (3). Expression (3) implies expression (4) by induction; below is the step to show that (3) implies (4) for $n = 3$: For matching M decomposed into three positive matchings M_1, M_2, M_3,

$$\begin{aligned}
I(M) &= \frac{|M_1|}{|M_1|+|M_2+M_3|} I(M_1) + \frac{|M_2+M_3|}{|M_1|+|M_2+M_3|} I(M_2+M_3) \\
&= \frac{|M_1|}{|M_1|+|M_2|+|M_3|} I(M_1) + \frac{|M_2+M_3|}{|M_1|+|M_2|+|M_3|} \left[\frac{|M_2|}{|M_2|+|M_3|} I(M_2) + \frac{|M_3|}{|M_2|+|M_3|} I(M_3)\right] \\
&= \frac{|M_1|}{|M_1|+|M_2|+|M_3|} I(M_1) + \frac{|M_2|}{|M_1|+|M_2|+|M_3|} I(M_2) + \frac{|M_3|}{|M_1|+|M_2|+|M_3|} I(M_3).
\end{aligned}$$

affine transformation, that satisfies population decomposability and the basic invariance and monotonicity axioms. Under population decomposability, Claims 1 and 4 combine to imply the equivalence of the two notions of monotonicity—diagonal and off-diagonal monotonicity and marginal monotonicity—and so the characterization result holds for both notions of monotonicity.

Theorem 2. *An index satisfies the basic axioms (i.e., of invariance, monotonicity, and homogamy) and population decomposability if and only if it is a positive affine transformation of normalized trace.*

Equivalently, the normalized trace is the unique index, up to positive affine transformation, that satisfies the three invariance axioms, marginal monotonicity, maximum homogamy, and population decomposability. Or alternatively, it is the unique index, up to positive affine transformation, that satisfies the three invariance axioms, diagonal monotonicity, off-diagonal monotonicity, maximum homogamy, and population decomposability.

In addition, because of uniqueness up to a positive affine transformation, the index satisfying all basic axioms and population decomposability is uniquely determined if the range is specified. For example, when the range is $[0, 1]$, the unique index is the normalized trace. Maximum homogamy is used to pin down the boundary cases of maximally but not fully positive-assortative matchings, i.e., (a, b, c, d) when $b = 0$ or $c = 0$ (but not both). We restricted the normalized trace of a matching of form $(a, b, 0, d)$, where $abd > 0$, to be 1. If we drop maximum homogamy as a criterion, then the normalized trace without this restriction—namely, $I(a, b, c, d) = (a+d)/(a+b+c+d)$ for any matching matrix (a, b, c, d)—also satisfies the other aforementioned properties.

4.3 Random decomposability and aggregate likelihood ratio

The aggregate likelihood ratio is characterized by a decomposability axiom in which the summation weights are the expected mass of pairs of like types under random matching.

Random Decomposability. For any positive matching matrices M and M',

$$I(M+M') = \frac{r(M)}{r(M+M')}I(M) + \frac{r(M')}{r(M+M')}I(M'),$$

where $r(M)$ indicates the expected mass of pairs of like types under random matching in market M:

$$r(M) \equiv \frac{a+b}{|M|}\frac{a+c}{|M|}|M| + \frac{d+b}{|M|}\frac{d+c}{|M|}|M| = \frac{(a+b)(a+c)+(d+b)(d+c)}{a+b+c+d}.$$

Equivalently, for any positive M_n, $n = 1, 2, \ldots, N$,

$$I\left(\sum_n M_n\right) = \sum_n \left[\frac{r(M_n)}{r\left(\sum_n M_n\right)} I(M_n)\right].$$

Theorem 3. *An index satisfies the three invariance axioms, marginal monotonicity, and random decomposability if and only if it is a positive multiple of aggregate likelihood ratio.*

We make two comments. First, note that the aggregate likelihood ratio does not satisfy maximum homogamy, diagonal monotonicity, or off-diagonal monotonicity, so we do not have a characterization result that connects those axioms and the aggregate likelihood ratio. Second, note that the class of measures that satisfy the axioms in Theorem 3 must be a positive multiple—i.e., a positive affine transformation without an intercept—of the aggregate likelihood ratio. This is because the weights $r(M)/r(M+M')$ and $r(M')/r(M+M')$ do not necessarily add up to be 1. Nonetheless, this characterization result gives the aggregate likelihood ratio a cardinal interpretation.

To address the observation that the weights do not necessarily add up to one, consider an axiom that involves weighted averages that add up to one and compares markets with proportional marginal distributions.

Marginal Random Decomposability. For all $M > \mathbf{0}$ and $M' > \mathbf{0}$ such that $M/|M|$ and $M'/|M'|$ have the same marginal distributions,

$$I(M + M') = \frac{r(M)}{r(M) + r(M')} I(M) + \frac{r(M')}{r(M) + r(M')} I(M').$$

Aggregate likelihood ratio satisfies this axiom, but it is not the unique measure that does so. For example, the normalized trace—and as a result, any linear combination of aggregate likelihood ratio and normalized trace—also satisfies this axiom.

5 Structural interpretation of the odds ratio

It is important to note that among the various indices, the odds ratio has a known structural interpretation. Specifically, assume that the observed matching behavior constitutes the stable equilibrium of a frictionless matching model under transferable utility. Assume, furthermore, that the surplus generated by a match between woman x belonging to category θ_i and man y belonging to category θ_j takes the separable form

$$s(x, y) = Z^{\theta_i \theta_j} + \alpha_x^{\theta_j} + \beta_y^{\theta_i}, \tag{5}$$

where Z is a deterministic component depending only on individual educations and the α's and β's are random shocks reflecting unobserved heterogeneity among individuals.[13] It is now well known (Graham, 2011; Chiappori, 2017) that, keeping constant the distribution of the shocks, assortativeness is related to the supermodularity of the matrix $Z^{\theta_i \theta_j}$, i.e., in the two-type case, to the sign of the *supermodular core* $Z^{\theta_1 \theta_1} + Z^{\theta_2 \theta_2} - Z^{\theta_1 \theta_2} - Z^{\theta_2 \theta_1}$. More importantly, if, following the seminal contribution by Choo and Siow (2006), one assumes that the random shocks follow type-1 extreme value distributions—the so-called *Separable Extreme Value* model—then the supermodular core equals twice the odds ratio. Hence, the

[13] Transferable-utility models satisfying condition (5) are referred to as separable models in the literature.

odds ratio represents the average gain per couple from positive-assortative matching over negative-assortative matching.

This structural interpretation is especially useful for disentangling possible changes in the value of different matches from the mechanical effect of variations in the marginal distributions of education among individuals. “Structural changes,” here, can only affect either the matrix Z or the distributions of random shocks. It is also useful for constructing counterfactual simulations, since the same structure can be applied to different distributions of education by genders, using standard techniques to solve for the stable equilibrium of the corresponding matching game.[14]

We make two remarks at this point. First, as clearly pointed out by Choo and Siow (2006) as well as by the subsequent literature (Galichon and Salanié, 2022; Chiappori and Salanié, 2016), this structural model can be identified (in the econometric sense) from matching patterns, but only under strong parametric restrictions. For instance, the initial Choo and Siow (2006) framework, which generates the odds ratio as an estimator of the supermodularity of the surplus, is *exactly* identified under the assumption that the random shocks α's and β's follow type-1 extreme value distributions. Indeed, the consensus is that, to be identified from the sole observation of matching patterns, any structural model would require strong parametric assumptions regarding the distribution of random preference shocks. It follows that, in general, the ranking (in terms of assortativeness, i.e., supermodularity of the deterministic surplus) may vary with the specific assumptions made on the distribution of the stochastic factors.

Several routes can be followed to overcome this limitation. Following Chiappori, Salanié, and Weiss (2017), one may consider repeated cross sections and impose restrictions on how the structural components change over time. Alternatively, Chiappori, Costa Dias, and Meghir (2018) argue that direct observation of post-marital behavior provides additional information on the surplus (since the latter, under transferable utility, is simply the sum

[14] See Chiappori et al. (2020) and Chiappori, Costa Dias, and Meghir (2020) for applications.

of individual utilities and can be estimated from the observation of the demand functions); this information can be used in particular to relax parametric assumptions made on the stochastic components of the model. In a recent contribution, Gualdani and Sinha (2023) also show how partial (i.e., set) identification may obtain using general assumptions on the distribution of stochastic shocks (for instance, independence of taste shocks from covariates and quantile, or symmetry restrictions).

Second and more importantly, while robust examples can be given where an assortativeness ranking is reversed when the assumptions regarding stochastic distributions are changed, one can nevertheless define conditions under which the ranking will be the same for any separable transferable utility model, *irrespective of the stochastic distribution*, provided the latter satisfy some basic properties (such as independence). Specifically, Chiappori, Costa Dias, and Meghir (2020) show that if two PAM M and M' are such that:

$$\frac{a}{a+b} \geq \frac{a'}{a'+b'}, \quad \frac{a}{a+c} \geq \frac{a'}{a'+c'}, \quad \frac{d}{b+d} \geq \frac{d'}{b'+d'} \quad \text{and} \quad \frac{d}{c+d} \geq \frac{d'}{c'+d'}, \tag{6}$$

then irrespective of the stochastic distributions of α and β (provided they are independent from each other and from the observed characteristics), the deterministic surplus corresponding to M will be more supermodular than M'; this is the *Generalized Separable (GS)* criterion in Chiappori, Costa Dias, and Meghir (2020). As a result, for any stochastic distributions of α's and β's, while the numerical value of the supermodular cores will depend on the choice of distributions, the structural model *will always rank M above M' in terms of assortativeness.*

In other words, one can define a preorder that is totally robust to changes in distributional assumptions. The price to pay for this generalization, however, is that the preorder is no longer complete: If some inequalities in (6) are satisfied while others are violated, the matching matrices simply cannot be compared.

6 Multiple types

There are 14 different education categories in the census, for example. It is worthwhile to consider how to compare the assortativeness of matching matrices of more than two types (if we must compare). Consider matching $M = (\mu_{ij})_{i,j \in \{1,\cdots,n\}}$ for $n \geq 3$.

6.1 Extensions from two types

The aggregate likelihood ratio and normalized trace can be naturally extended to multi-type markets, but the odds ratio does not have a natural extension. The *normalized trace for multi-type markets* is

$$I_{tr}(M) = \begin{cases} 1 & \text{if } \mu_{ij}\mu_{ji} = 0 \ \forall i \text{ and } j \neq i \\ 0 & \text{if } \mathrm{tr}(M) \equiv \sum_{i=1}^{n} \mu_{ii} = 0 \\ \mathrm{tr}(M)/|M| & \text{otherwise.} \end{cases}$$

The *aggregate likelihood ratio for multi-type markets* is

$$I_L(M) \equiv \frac{tr(M)/|M|}{\sum_{i=1}^{n} \left(\frac{\sum_{j=1}^{n} \mu_{ij}}{|M|}\right)\left(\frac{\sum_{j=1}^{n} \mu_{ji}}{|M|}\right)} = \frac{|M|(\sum_{i=1}^{n} \mu_{ii})}{\sum_{i=1}^{n}(\sum_{j=1}^{n} \mu_{ij})(\sum_{j=1}^{n} \mu_{ji})}.$$

Theorems 2 and 3 can be naturally extended so that normalized trace and the aggregate likelihood ratio are the unique indices that satisfy the sets of axioms (appropriately modified) in those theorems. Hence, normalized trace and the aggregate likelihood ratio can still be used to measure the assortativeness of multi-type markets. In addition, we can extend the measures to measure assortativeness in (i) markets with unmarried and (ii) one-sided (e.g., homosexual) matching markets. See Zhang (2024) for more detailed theoretical discussions and empirical analyses of these markets.

6.2 Robustness to categorization

Consider three types θ_i, θ_j, and θ_k. When two types θ_i and θ_j merge so that the three categories are partitioned to $\pi = \{\{\theta_i, \theta_j\}, \{\theta_k\}\}$, the matching given this categorization becomes

$$M^\pi = \begin{pmatrix} \mu_{ii} + \mu_{ij} + \mu_{ji} + \mu_{jj} & \mu_{ik} + \mu_{jk} \\ \mu_{ki} + \mu_{kj} & \mu_{kk} \end{pmatrix}.$$

We say that M_1 is more assortative than M_2 if and only if M_1^π is more assortative than M_2^π for all partitions π. For example, with normalized trace, for M and M' such that $|M| = |M'|$ and $\text{tr}(M) = \text{tr}(M')$, M is more assortative than M' if and only if $\mu_{12} + \mu_{21} \geq \mu'_{12} + \mu'_{21}$, $\mu_{13} + \mu_{31} \geq \mu'_{13} + \mu'_{31}$, and $\mu_{23} + \mu_{32} \geq \mu'_{23} + \mu'_{32}$. Formally, we define *robustness to categorization* as follows.

Robustness to Categorization. Let categorization Π denote the set of partitions of types to be considered. Let M^π denote the matching under partition $\pi \in \Pi$. A preorder $\succeq$ is robust to categorization Π when for all matrices M_1 and M_2, $M_1 \succeq M_2$ if and only if $M_1^\pi \succeq M_2^\pi$ for any partition $\pi \in \Pi$, and $M_1 \succ M_2$ if and only if $M_1^\pi \succ M_2^\pi$ for any partition $\pi \in \Pi$.

Theorem 4. *Let* Π *be a nondegenerate categorization. No total preorder satisfies marginal monotonicity and robustness to categorization* Π*.*

A counterexample suffices for the claim. Although the counterexample is one of three-type matching matrices, it suffices for the claim to be valid for matching matrices with more than three types.

Proof of Theorem 4. Consider matrices M and M_ϵ of three types θ_1, θ_2, and θ_3,

$$M = \begin{array}{c|c|c} 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \end{array} \text{ and } M_\epsilon = \begin{array}{c|c|c} 1/9-\epsilon & 1/9+\epsilon & 1/9 \\ \hline 1/9+\epsilon & 1/9 & 1/9-\epsilon \\ \hline 1/9 & 1/9-\epsilon & 1/9+\epsilon \end{array} = M + \begin{array}{c|c|c} -\epsilon & +\epsilon & 0 \\ \hline +\epsilon & 0 & -\epsilon \\ \hline 0 & -\epsilon & +\epsilon \end{array}.$$

Under partition $\pi_{12} = \{\{\theta_1, \theta_2\}, \{\theta_3\}\}$, we group θ_1 and θ_2,

$$M^{\pi_{12}} = \begin{array}{c|c} 4/9 & 2/9 \\ \hline 2/9 & 1/9 \end{array} \text{ and } M_\epsilon^{\pi_{12}} = \begin{array}{c|c} 4/9+\epsilon & 2/9-\epsilon \\ \hline 2/9-\epsilon & 1/9+\epsilon \end{array} = M^{\pi_{12}} + \begin{array}{c|c} +\epsilon & -\epsilon \\ \hline -\epsilon & +\epsilon \end{array}.$$

Under partition $\pi_{23} = \{\theta_1, \{\theta_2, \theta_3\}\}$, we group θ_2 and θ_3,

$$M^{\pi_{23}} = \begin{array}{c|c} 1/9 & 2/9 \\ \hline 2/9 & 4/9 \end{array} \text{ and } M_\epsilon^{\pi_{23}} = \begin{array}{c|c} 1/9-\epsilon & 2/9+\epsilon \\ \hline 2/9+\epsilon & 4/9-\epsilon \end{array} = M^{\pi_{23}} + \begin{array}{c|c} -\epsilon & +\epsilon \\ \hline +\epsilon & -\epsilon \end{array}.$$

By marginal monotonicity,

$$M^{\pi_{12}} \prec M_\epsilon^{\pi_{12}} \text{ and } M^{\pi_{23}} \succ M_\epsilon^{\pi_{23}}.$$

Hence, there does not exist a total preorder that satisfies marginal monotonicity and robustness to categorization. □

The impossibility result suggests that we must resort to partial rankings to satisfy some notion of monotonicity and robustness to categorization. Existing partial rankings such as supermodular stochastic order (Meyer and Strulovici, 2012, 2015) and positive quadrant dependence order (Anderson and Smith, 2024) are natural candidates. However, they only compare matching matrices of the same marginals. Hence, further investigations to compare matching matrices of differential marginal distributions are needed.

7 Educational homogamy in the US

Our paper was in part motivated by the conflicting results on how educational homogamy has changed in the US. We now consider its evolution for recent cohorts, through the lens of our discussion. Our goal is to compare and contrast the answers given by various indices to the same basic question: Did educational homogamy increase between different cohorts? Gihleb and Lang (2020) consider this issue, using a variety of indices (correlation, rank correlation, Goodman and Kruskal's γ, Kendall's τ adjusted and not adjusted for ties). A difference from our approach is that they consider changes affecting the entire distribution of education and many classes, whereas we in addition consider how assortativeness changed locally at the top of the education distribution using our 2×2 framework.

We use the March extract of the US Current Population Survey, from where we select the subsample of married individuals (including legal marriage and cohabitation) observed when aged 35-44 and belonging to the 1950s or the 1970s cohorts.[15] This is an interesting comparison to make because just after the 50's cohort educational attainment of women accelerated and overtook that of men, which also increased but less so.(Chiappori, Iyigun, and Weiss, 2009; Zhang, 2021). Relatedly, the returns to investments in human capital increased substantially during the 1980s and later (Katz and Murphy, 1992). Educational and marital choices by the older 1950s generation were mostly made before that period; in contrast, individuals born in the 1970s, when choosing both their education level and their spouse, were fully aware of the new context. Consequently, it has been argued that homogamy should have increased between these two cohorts (for instance, Chiappori, 2017; Chiappori, Salanié, and Weiss, 2017), which is what we reconsider here. Specifically, we define the 1950s cohort as married couples where at least one partner was born in the 1950-59 period and similarly for the 1970s cohort.

Following Gihleb and Lang (2020), who highlight the sensitivity of empirical changes

[15] In Appendix Table 5 we present comparisons of the 70's with other cohorts, which also show interesting patterns.

in assortativeness to separating college graduates and post-graduates over this period, we consider 5 education groups: high-school dropouts (HSD), high-school graduates (HS), some college education but no degree (SC), 4-year college degree (C) and post-graduate degree (meaning more than bachelor's degree, PG).

Table 2: Marginal distributions of education and matching in the 1950s and the 1970s cohorts

Category	**HSD**	**HS**	**SC**	**C**	**PG**	**Marg Dist: Men**
			1950s Cohort			
Men			**Women**			
HSD	4.7	3.9	1.1	0.2	0.0	9.9
HS	3.1	20.0	7.1	2.4	0.6	33.2
SC	0.9	9.1	10.8	3.8	1.0	25.6
C	0.2	3.9	5.8	7.8	2.0	19.6
PG	0.1	1.1	2.4	4.4	3.7	11.6
Marg Dist: Women	8.9	37.9	27.2	18.6	7.4	
			1970s Cohort			
Men			**Women**			
HSD	4.5	2.4	1.2	0.4	0.1	8.6
HS	2.0	13.5	7.8	3.8	1.3	28.4
SC	0.6	4.9	11.5	5.9	2.4	25.3
C	0.2	2.0	4.8	11.6	4.9	23.4
PG	0.1	0.5	1.8	5.3	6.6	14.3
Marg Dist: Women	7.3	23.4	27.0	27.0	15.3	

Educational categories: HSD: High School dropout, HS: High School, SC: Some College, C: College, PG: Post-graduate. Numbers represent cell percentages.

Table 2 shows the marginal distributions of education for men and women and the resulting matching patterns. Both panels show evidence of positive sorting, with most of the mass concentrated on the main diagonal. They also show how women with college or more education increased dramatically between these two cohorts, making this group larger than the corresponding one for men, which also increased but not by nearly as much. These changes in the marginal distributions are at the heart of the contradictory results of the various measures. The question is whether sorting became more positive between the two cohorts.

Table 3 presents the results. For each index and each matching (sub)matrix, we present its change across the two cohorts, with a positive number indicating an increase in positive sorting. Since we are testing differences across many indices and many matching matrices, we adjust our p-values for multiple testing using the Romano-Wolf stepdown approach (Romano and Wolf, 2005). The table presents two sets of p-values: in round brackets are the most conservative, allowing for all 30 estimates; in square brackets are p-values adjusted for multiple testing within each submarket only.

Table 3: Marital assortativeness – comparing cohorts born between 1950-59 and 1970-75

		Submarkets			Global indices	
		C&PG vs SC	PG vs C	C vs SC	C&PG vs others	5 educ levels
		(1)	(2)	(3)	(4)	(5)
(1)	Odds ratio	1.173	-0.400	0.920	0.639	
		(.00) [.00]	(.04) [.01]	(.00) [.00]	(.11) [.02]	
(2)	L1 ratio (high ed)	-0.046	-0.074	-0.025	-0.421	
		(.00) [.00]	(.00) [.00]	(.10) [.02]	(.00) [.00]	
(3)	L2 ratio (low ed)	0.246	0.011	0.113	0.155	
		(.00) [.00]	(.32) [.21]	(.00) [.00]	(.00) [.00]	
(4)	Weighted L ratio	-0.013	-0.003	0.058	0.149	0.223
		(.17) [.09]	(.95) [.78]	(.00) [.00]	(.00) [.00]	(.00) [.00]
(5)	Normalized trace	0.041	-0.004	0.024	-0.017	0.007
		(.00) [.00]	(.76) [.55]	(.00) [.00]	(.00) [.00]	(.07) [.01]
(6)	χ^2	0.029	-0.008	0.035	0.040	
		(.00) [.00]	(.31) [.20]	(.00) [.00]	(.00) [.00]	
(7)	Minimum distance	0.001	-0.099	0.028	0.026	
		(.88) [.88]	(.00) [.00]	(.04) [.01]	(.00) [.00]	

Notes: Education attainment in 5 groups: post-graduate degrees (PG, more than 4-years college), 4-years college degrees (C), some college education but no degree (SC), high school qualifications (HS), and no qualifications (HSD). First column refers to matching matrices between graduates (graduates and post-graduates taken together) and those with some college education but not a degree; column 2: post-graduates and 4-year college graduates; column 3: college graduates and some college; column 4: college graduates and everyone else together; column 5: all 5 education groups separate. For each index, the top row shows estimates of the difference in the respective index between the latest and earliest cohort; the number in round brackets shows p-values for 2-sided significance testing adjusted for multiple hypothesis using the stepdown method jointly for all the 30 measures on the table (Romano and Wolf, 2005; Romano, Shaikh, and Wolf, 2008; Romano and Wolf, 2016); the number in square brackets shows stepdown p-values but for each market (column) separately. *Data source*: March extract of the US Current Population Survey, subsample of legally married and cohabiting individuals observed when aged 35-44 and born in 1950-59 and 1970-75.

Looking first at estimates in column 5 for the whole disaggregated 5×5 sorting matrix, both the normalized trace and the aggregate likelihood ratio (the two indices that are designed for such a comparison) imply an overall increase in PAM, highly significant in the case of the likelihood ratio but only marginally so at conventional significance levels for the normalized trace. The change in the normalized trace is also close to zero, pointing to an increase of 0.7 percentage points in the mass of homogamous marriages across the two cohorts.[16] On the basis of these two measures, sorting increased or remained the same between these two cohorts.

Column 4 also displays estimates for the entire population of couples, but aggregating education in two larger groups: college-educated (graduates and post-graduates), and all those who did not gain college qualifications, so that the comparison is between two 2×2 matrices. Under this aggregation, the aggregate likelihood ratio still shows a strongly significant increase in assortativeness, while the normalized trace shows a strongly significant drop. The latter confirms the conclusion in Gihleb and Lang (2020) that aggregation matters. Taken together, even measures satisfying the basic axioms (of invariance, monotonicity, homogamy) may point in opposite directions when comparing assortativeness in two matching matrices. In some instances, the ambiguity can be resolved by requiring that measures satisfy some characterization axiom, such as marginal independence, which leaves the odds ratio as the uniquely acceptable index. However, the impossibility result in Theorem 4 points to the limitations.

Consider now the likelihood ratios. In all columns 1-4 involving comparisons of 2×2 matrices, the two type-specific likelihood ratio indices differ by whether the basis for its construction is the matching of the high education types with each other (L1 in row 2, type θ_1) or the low education types with each other (L2 in row 3, type θ_2). These conflicting findings

[16]Note that the different indices vary in scale, so comparing the size of their changes is not meaningful. Most indices, including the odds ratio, lack a cardinal interpretation, and hence comparing how much they change across different markets is also not meaningful. However, the normalized trace stands out for having a cardinal interpretation, implied by its population decomposability property, and hence the size of changes in this measure across markets can be interpreted and compared.

between the two type-specific likelihood ratios are possible because they do not satisfy the basic axiom of type invariance. The aggregate likelihood ratio is a weighted average of the two, and depending on the relative mass associated with each, it can sometimes be positive or negative. Moreover, none of the likelihood ratios satisfies diagonal monotonicity or the homogamy axioms. This implies that in cases where there is a natural ranking of assortativeness revealed unambiguously by all other indices we examined (such as when one of the markets is fully homogamous, or when two markets differ by the mass in a single cell), it is possible that the likelihood ratios will show contrary findings. More generally, failure to satisfy these properties can result in the likelihood ratios frequently finding changes in assortativeness in the opposite direction of other measures that do satisfy them.

Submarket comparisons. Assortativeness can change differently, even in opposite directions, for different parts of the education distribution, making the overall indices difficult to interpret. Local differences in how much PAM changes across two matching matrices may also contribute to ambiguity in findings related to whole markets, as different measures weigh the various margins differently. Given the stark differences in university graduation rates across the two cohorts, we now focus on sorting at the top of the education distribution, between college-educated and those who attend for some time but drop out of college without a degree.

Column 1 considers the submarket of those with some college degree (C or PG) to those with some college attendance, but no 4-year degree (SC). Setting aside the likelihood ratios, all indices in column 1 show either a significant increase in sorting or no (significant) change at all (minimum distance). As we noted above, the direction of the type-specific likelihood ratios depends on how it is computed (L1 or L2), while the aggregate likelihood ratio points to a reduction in assortativeness but is not statistically significant at conventional levels.

To gain further insight, we separate those with a college degree by whether or not they obtained post-graduate degrees. Columns 2 and 3 show results for PG versus C and for

C versus SC respectively. While the indices consistently point to a significant increase in assortativeness between college graduates and some college (column 3), assortativeness may actually have decreased among graduates and post-graduates (column 2), though estimates are only statistically significant for the odds ratio and the minimum distance. The opposing changes in these two submarkets can help explain the ambiguous results in the literature, which most refer to as global measures of assortativeness.

Table 4: Testing for assortativeness based on the structural model with no distributional assumptions, cohorts 1970s versus 1950s

		Submarkets			Global index
		C&PG vs SC	PG vs C	C vs SC	C&PG vs others
		(1)	(2)	(3)	(4)
(1)	$\Delta\frac{a}{a+b}$	0.126	-0.097	-0.077	0.181
		(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]
(2)	$\Delta\frac{a}{a+c}$	-0.013	0.045	0.056	-0.018
		(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]
(3)	$\Delta\frac{d}{d+b}$	0.069	-0.078	-0.010	0.020
		(.00) [.00]	(.00) [.00]	(.14) [.14]	(.00) [.00]
(4)	$\Delta\frac{d}{d+c}$	-0.109	0.097	0.133	-0.105
		(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]

Notes: Δ signifies change from the 1950s to the 1970s cohort. Education attainment separates post-graduate degrees (PG more than 4-years college), 4-years college degrees (C), some college education but no degree (SC), and all others (high school qualifications HS, and no qualifications HSD). For each index, the top row shows estimates of the difference in the respective index between the latest and earliest cohort; the number in round brackets shows p-values for 2-sided significance testing adjusted for multiple hypothesis using the stepdown method jointly for all the 16 measures on the table (Romano and Wolf, 2005; Romano, Shaikh, and Wolf, 2008; Romano and Wolf, 2016); the number in square brackets shows stepdown p-values but for each market (column) separately. *Data source*: March extract of the US Current Population Survey, subsample of legally married and cohabiting individuals observed when aged 35-44 and born in 1950-59 and 1970-75.

Identifying changes in sorting based on a distribution-free structural approach. We can apply our generalized separable conditions (6) to examine whether sorting increased or not. Table 4 shows estimates of the differences in the respective ratios. Identifying unambiguous changes in assortativeness across two markets requires that all differences in a column have the same sign: positive for an increase in PAM, and negative for a reduction. As shown in the table, the conditions for increased sorting under this criterion are not all simultaneously satisfied for any of the markets considered. This implies that, for the markets studied here, one can always find specific distributions for random preference terms in the separable transferable utility model such that the ranking would be reversed, a conclusion similar to that of Gualdani and Sinha (2023). While imposing marginal independence is attractive, we note that the odds ratio is not decomposable, while the normalized trace is (but does not satisfy marginal independence).

Discussion. Setting aside the likelihood ratios, all other indices point to an increase in assortativeness at the top of the education distribution with the exception of the very top, in the submarket involving those with post-graduate degrees matching with those with just 4-year college (bachelor's degree). The single exception is that of the normalized trace, which shows a drop in assortativeness when the entire market is aggregated in two big classes, contradicting its own predictions when assortativeness at the top is investigated at a more local level.

In other settings, however, there could be more ambiguity than we observe here, even between indices that satisfy all the basic axioms. In such cases, it will be especially important to consider the additional characterization properties that could help resolve the ambiguity. When comparing the markets for the two cohorts (1950s and 1970s), large changes in the marginal distributions stand out, especially for women for whom college attainment increased substantially. In cases such as this one, imposing marginal independence seems especially relevant: Changes in assortativeness relate to how the additional 1970s college women match with high versus low educated men in proportions similar or different to those for the college women in the 1950s. Imposing marginal independence means adopting the odds ratio, which under the Separable Extreme Value model admits an economic interpretation based on optimal partner choice and marriage market equilibrium. However, a question that remains open and in need of further investigation is the characterization of the complete class of structural economic models that satisfy the basic axioms and marginal independence. Discovering this, which is beyond the scope of this paper, will provide the complete link between the axiomatic approach and one based on optimizing economic behavior and the equilibrium in the marriage market.

8 Concluding Remarks

It is relatively simple to estimate whether there is positive assortative matching in a stochastic marriage market along the dimensions of a characteristic such as education. However, measuring the extent to which such assortative matching differs between two economies or between two points in time for the same economy is challenging when the marginal distributions of the characteristics also change.

In this paper, we show that different measures may generate different conclusions regarding the evolution of educational homogamy over time. We first take an axiomatic approach to better understand the underlying properties of commonly used measures. Namely, we axiomatize the odds ratio, highlighting the marginal independence property. We discuss a structural interpretation of the odds ratio. In addition, we also discuss the difficulties of comparing assortativeness for markets with more than two types. We then revisit the changes in assortative matching in the US between the 50's and the 70's cohort, between which there were large changes in the educational attainment of men and even more so for women. The odds ratio points to an increase in sorting with the exception of the very top of the education distribution that involves post-graduate degrees.

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A Online Appendix

A.1 Omitted proofs for characterization results

Proof of Claim 2. Suppose $M = (a, b, c, d)$ and $M' = (a', b', c', d')$ have the same marginal distributions and $M \succ M'$. By marginal independence,

$$\begin{aligned}(a,b,c,d) &\sim (a,b,c\frac{c'}{c},d\frac{c'}{c}) \sim (a,b\frac{c}{c'}\frac{d'}{d},c',d\frac{c'}{c}\frac{c}{c'}\frac{d'}{d}) \sim (a\frac{b'}{b}\frac{c'}{c}\frac{d}{d'},b\frac{b'}{b}\frac{c'}{c}\frac{d}{d'}\frac{c}{c'}\frac{d'}{d},c',d') \\ &\sim (a'\frac{a}{a'}\frac{b'}{b}\frac{c'}{c}\frac{d}{d'},b',c',d').\end{aligned}$$

Let $\delta \equiv \frac{a}{a'}\frac{b'}{b}\frac{c'}{c}\frac{d}{d'}$. By marginal monotonicity, $a > a'$, $b < b'$, $c > c'$, and $d > d'$. Hence, $\delta > 1$. The relations $(a\delta, b', c', d') \sim (a', b'/\delta, c', d') \sim (a', b', c'/\delta, d') \sim (a', b', c', d'\delta) \succ (a', b', c', d')$ for all a', b', c', and d' imply diagonal monotonicity and off-diagonal monotonicity. □

Proof of Claim 3. Suppose preorder $\succeq$ satisfies marginal monotonicity. Then we have for any δ,

$$(a, b, c, d) \succ (a - \delta, b + \delta, c + \delta, d + \delta).$$

By (repeatedly applying) marginal independence,

$$\begin{aligned}(a-\delta,b+\delta,c+\delta,d+\delta) &\sim (a,b+\delta,(c+\delta)\frac{a}{a-\delta},d-\delta) \\ &\sim (a,b,(c+\delta)\frac{a}{a-\delta},(d-\delta)\frac{b}{b+\delta}) \\ &\sim (a,b,c,d\frac{a-\delta}{a}\frac{b}{b+\delta}\frac{c}{c+\delta}\frac{d-\delta}{d}) \equiv (a,b,c,d\lambda(\delta)),\end{aligned}$$

where $\lambda(\delta)$ is strictly smaller than 1 for any $\delta > 0$ and is continuously decreasing in δ. With similar transformations, we have

$$\begin{aligned}&(a-\delta,b+\delta,c+\delta,d+\delta) \\ \sim\ &(a,b,c,d\lambda(\delta)) \sim (a\lambda(\delta),b,c,d) \sim (a,b/\lambda(\delta),c,d) \sim (a,b,c/\lambda(\delta),d).\end{aligned}$$

Hence, for any $\delta > 0$, diagonal monotonicity holds:

$$(a, b, c, d) \succ (a, b, c, d\lambda(\delta)) \sim (\lambda(\delta)a, b, c, d),$$

and off-diagonal monotonicity holds:

$$(a, b, c, d) \succ (a, b/\lambda(\delta), c, d) \sim (a, b, c/\lambda(\delta), d).$$

Reversely,

$$(a, b, c, d) \succ (a, b, c, d\lambda(\delta)) \sim (\lambda(\delta)a, b, c, d)$$

for any δ implies

$$(a, b, c, d) \succ (a, b/\lambda(\delta), c, d) \sim (a, b, c/\lambda(\delta), d)$$

for any δ, and

$$(a, b, c, d) \succ I(a - \delta, b + \delta, c + \delta, d - \delta)$$

for any δ, where marginal independence is repeatedly used again. □

Proof of Theorem 1. It is straightforward to check that the preorder induced by the odds ratio satisfies marginal monotonicity, maximum homogamy, and marginal independence.

It remains to show that there does not exist an index I (or equivalently in the current context, a preorder induced by index I) that satisfies these axioms but is not preorder-equivalent to the odds ratio. Suppose by way of contradiction that such an index exists. Then, we must have for some $M = (a, b, c, d)$ and $M' = (a', b', c', d')$. One of the following four cases occurs: (i) $I(M) > I(M')$ and $Q(M) < Q(M')$, (ii) $I(M) < I(M')$ and $Q(M) > Q(M')$, (iii) $I(M) = I(M')$ and $Q(M) \neq Q(M')$, and (iv) $I(M) \neq I(M')$ and $Q(M) = Q(M')$.

First, suppose that case (i) $I(M) > I(M')$ and $Q(M) < Q(M')$ occurs. Because $I(M) > I(M')$, by the implication of DM, $ad \neq 0$, and by the implication of ODM, $b'c' \neq 0$, and because $Q(M) < Q(M')$, similarly, $bc \neq 0$ and $a'd' \neq 0$. For each of the following steps, we invoke a part of marginal independence:

$$
\begin{aligned}
I(a,b,c,d) &= I(\frac{b'}{b}a, \frac{b'}{b}b, c, d) \\
&= I(\frac{b'}{b}a\frac{a'}{a}\frac{b}{b'}, b', c\frac{a'}{a}\frac{b}{b'}, d) \\
&= I(a', b', c\frac{a'}{a}\frac{b}{b'} \cdot \frac{c'}{c}\frac{a}{a'}\frac{b'}{b}, d \cdot \frac{c'}{c}\frac{a}{a'}\frac{b'}{b}\frac{d'}{d'}) \\
&= I(a', b', c', d' \cdot \frac{ad}{bc}/\frac{a'd'}{b'c'}).
\end{aligned}
$$

By premise,

$$I(a', b', c', d' \cdot \frac{ad}{bc}/\frac{a'd'}{b'c'}) > I(a', b', c', d').$$

By diagonal monotonicity, this implies

$$\frac{ad}{bc} > \frac{a'd'}{b'c'}.$$

However, this implies $Q(M) > Q(M')$, which contradicts the premise that $Q(M) < Q(M')$.

For each of the four possibilities, using the same logic as above, a contradiction can be derived for any positive matchings M and M'.

Suppose there is a cell that is zero. Suppose $I(M) = I(M')$. If $bc = 0$, then $I(M) = I(a, 0, 0, d) = I(M')$, then $b'c' = 0$. In this case, by DM, $Q(M) = Q(M')$. If $ad = 0$ instead, then $I(M) = I(0, b, c, 0) = I(M')$ implies $a'd' = 0$. In this case, by ODM, $Q(M) = Q(M')$. Hence, whenever there is a cell with zero in one of the matrices, $I(M) = I(M')$ and $Q(M) = Q(M')$, which prevents all four cases from happening. □

Proof of Claim 4. Consider $M = (a, b, c, d) > \mathbf{0}$ and $M' = (a', b, c, d) > \mathbf{0}$, where $a' > a$. We want to show that $I(M') > I(M)$. By type invariance,

$$I(a, b, c, d) = I(d, c, b, a).$$

By population decomposability,

$$\frac{1}{2}I(a,b,c,d)+\frac{1}{2}I(d,c,b,a)=I\left(\frac{1}{2}(a+d),\frac{1}{2}(b+c),\frac{1}{2}(b+c),\frac{1}{2}(a+d)\right).$$

By scale invariance,

$$I(a,b,c,d)=I\left(\frac{a+d}{a+b+c+d},\frac{b+c}{a+b+c+d},\frac{b+c}{a+b+c+d},\frac{a+d}{a+b+c+d}\right).$$

By the same sequence of arguments by population decomposability, scale invariance, and type invariance,

$$I(a',b,c,d)=I\left(\frac{a'+d}{a'+b+c+d},\frac{b+c}{a'+b+c+d},\frac{b+c}{a'+b+c+d},\frac{a'+d}{a'+b+c+d}\right).$$

Note that the two matrices on the right-hand side of the two equations above have the same marginals (each row or column sums to 1). By marginal monotonicity, $a' > a$ implies

$$I(M)=I(a',b,c,d)>I(M')=I(a,b,c,d).$$

Hence, diagonal monotonicity is proved. Off-diagonal monotonicity can be shown analogously. □

Proof of Theorem 2. We first show that any index I that satisfies scale, side and type invariance, as well as diagonal and off-diagonal monotonicity, and population decomposability is order-equivalent to—i.e., a monotonic transformation of—normalized trace. Consider $M=(a,b,c,d)>\mathbf{0}$ and $M'=(a',b',c',d')>\mathbf{0}$. Here, we prove a lemma.

Lemma 1. *Any index that satisfies scale, side and type invariance, as well as diagonal and off-diagonal monotonicity, and population decomposability is an increasing function of $a+d$ and a decreasing function of $b+c$.*

Proof of Lemma 1. Consider $M=(a,x,x,d)$, and consider $M_1=(a/2,x-\lambda/2,\lambda/2,d/2)$

and $M_2 = (a/2, \lambda/2, x - \lambda/2, d/2)$ for some $\lambda \in (0, 2x)$. Note that $M_1 + M_2 = M$ and M_1 and M_2 have the same total mass. Hence, by population decomposability,

$$I(M) = \frac{1}{2} I(M_1) + \frac{1}{2}(M_2).$$

By side invariance, $I(M_1) = I(M_2)$. Hence, $I(M) = I(M_1) = I(M_2)$. By scale invariance,

$$I(M_1) = I(2M_1) = I(a, 2x - \lambda, \lambda, d).$$

Hence, for all $\lambda \in (0, 2x)$,

$$I(a, 2x - \lambda, \lambda, d) = I(a, x, x, d).$$

Hence, we have shown that $I(a, b, c, d) = I(a, b', c', d)$ whenever $b + c = b' + c'$. By the same logic, and by side invariance and type invariance, $I(a, b, c, d) = I(a', b, c, d')$ whenever $a + d = a' + d'$. I, by diagonal monotonicity, is strictly increasing in $a + d$ whenever $bc \neq 0$, and, by off-diagonal monotonicity, is strictly decreasing in $b + c$ whenever $ad \neq 0$. □

If (i) $a + d > a' + d'$ and $b + c \leq b' + c'$ or (ii) $a + d \leq a' + d'$ and $b + c > b' + c'$, then, by Lemma 1, (i) $I(M) > I(M')$ or (ii) $I(M) < I(M')$, respectively. Suppose $a + d > a' + d'$ and $b + c > b' + c'$. Define

$$M'' = (a'', b'', c'', d'') = M' \cdot (b + c)/(b' + c').$$

By definition of M'', $b'' + c'' = b + c$, and $a'' + d'' = (a' + d') \cdot (b + c)/(b' + c')$. By scale invariance of I, $I(M'') = I(M')$. The comparison of $a + d$ and $a'' + d''$ pins down the ordinal assortativeness relation between M and M'. That is,

$$\frac{a + d}{a'' + d''} = \frac{a + d}{b + c} \Big/ \frac{a' + d'}{b' + c'} > 1 \Leftrightarrow I(M) > I(M').$$

When $a + d < a' + d'$ and $b + c < b' + c'$, we can similarly pin down the ordinal assortativeness relation between M and M'. Note that for any $M = (a, b, c, d) > \mathbf{0}$,

$$I_{tr}(M) = \frac{(a+d)}{(a+b)+(c+d)} = \frac{a+d}{b+c} \Big/ \left(\frac{a+d}{b+c} + 1\right).$$

Hence, $I(M) > I(M')$ if and only if $I_{tr}(M) > I_{tr}(M')$.

Any nonlinear transformation of normalized trace would violate population decomposability, so any index that satisfies the stated axioms must be not only a monotonic transformation but also a positive affine transformation of normalized trace. □

Proof of Theorem 3. It is straightforward to check that the aggregate likelihood ratio satisfies the axioms, so it remains to show the other direction. We first show that any index I that satisfies all three invariance axioms, as well as marginal monotonicity, and random decomposability is proportional to the aggregate likelihood ratio. Consider $M = (a, b, c, d)$. By type invariance,

$$I(M) = I\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I\begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{7}$$

Recall

$$r(M) \equiv \frac{a+b}{|M|} + \frac{a+c}{|M|} + \frac{d+b}{|M|}\frac{d+c}{|M|}|M| = \frac{(a+b)(a+c)+(d+b)(d+c)}{a+b+c+d}.$$

By random decomposability,

$$I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} \cdot r\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} = I\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot r\begin{pmatrix} a & b \\ c & d \end{pmatrix} + I\begin{pmatrix} d & c \\ b & a \end{pmatrix} \cdot r\begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

which, by (7), is simplified to

$$I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} \frac{|M|^2 + |M|^2}{2|M|} = 2I(M)r(M) \Rightarrow I(M) = \frac{1}{2}\frac{|M|}{r(M)} I\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix}. \tag{8}$$

Because for any $\epsilon < \min\{a+d, b+c\}$,

$$\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} = \begin{pmatrix} a+d-\epsilon & \epsilon \\ \epsilon & a+d-\epsilon \end{pmatrix} + \begin{pmatrix} \epsilon & b+c-\epsilon \\ b+c-\epsilon & \epsilon \end{pmatrix},$$

by random decomposability, for any $\epsilon < \min\{a+d, b+c\}$,

$$|M| I \begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} = (a+d) I \begin{pmatrix} a+d-\epsilon & \epsilon \\ \epsilon & a+d-\epsilon \end{pmatrix} + (b+c) I \begin{pmatrix} \epsilon & b+c-\epsilon \\ b+c-\epsilon & \epsilon \end{pmatrix}.$$

Plugging in the expression of $I(M)$ in (8), we get, for any $\epsilon < \min\{a+d, b+c\}$,

$$I(M) = \frac{1}{2r(M)} \left[(a+d) \cdot I \begin{pmatrix} a+d-\epsilon & \epsilon \\ \epsilon & a+d-\epsilon \end{pmatrix} + (b+c) \cdot I \begin{pmatrix} \epsilon & b+c-\epsilon \\ b+c-\epsilon & \epsilon \end{pmatrix} \right].$$

Take $\epsilon \to 0^+$ and by scale invariance and $I(0,1,1,0) = 0$, we have

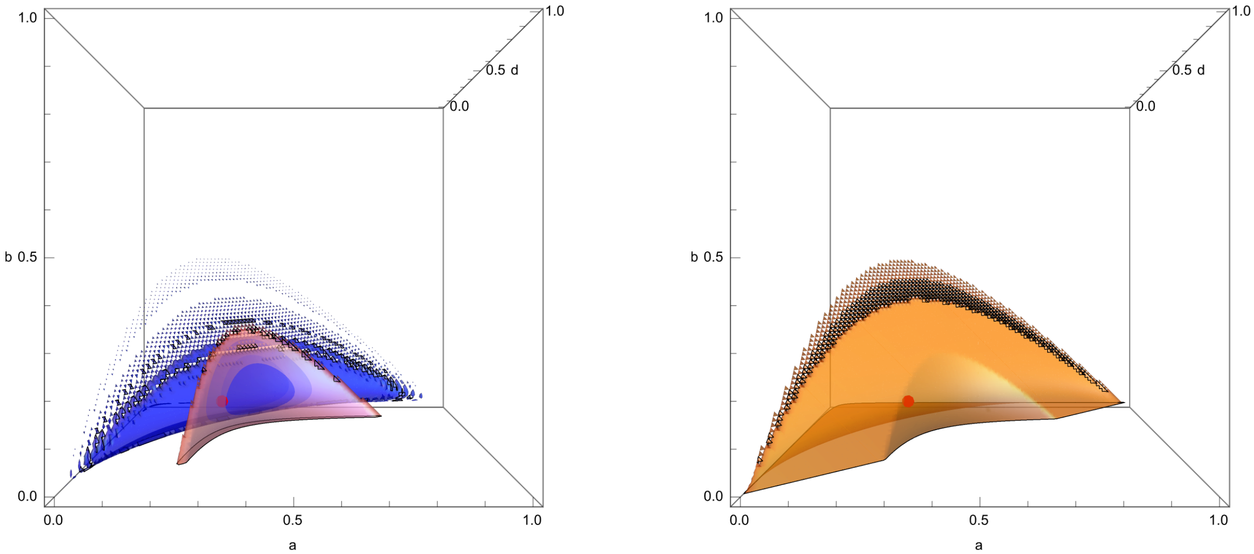
$$I(M) = \frac{1}{2} \frac{a+d}{r(M)} I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, any index that satisfies the above axioms is proportional to $(a+d)/r(M)$, the aggregate likelihood ratio. □

A.2 Iso-assortative surfaces

Fix a reference matching, $M = (0.3, 0.1, 0.1, 0.5)$ and consider markets with a unit mass of couples, $|M'| = 1$. Figure 2a shows what we call *iso-assortative surfaces* in the (a, b, d) three-dimensional space: The red iso-assortative surface collects the matching matrices $(a, b, 1-a-b-d, d)$ that are equally assortative to M according to the aggregate likelihood ratio, and the blue iso-assortative surface collects the matching matrices that are equally assortative to M according to the odds ratio. Because the iso-assortative surfaces are different, the

Figure 2: Iso-assortative surfaces and conflicting conclusions



(a) Iso-assortative surface

(b) Conflicting region

matching matrices that are judged to be more assortative than M by the two measures will differ. The orange region in Figure 2b consists of the matching matrices that are more assortative than M under the aggregate likelihood ratio but are less assortative than M under the odds ratio. Figure 1 illustrates *iso-assortative curves* and conflicting regions between the aggregate likelihood ratio and the odds ratio in a plane, by restricting the attention to matching matrices in which $b = c$. Mathematically speaking, Figure 1 illustrates the sliced plane of $b = (1 - a - d)/2$ in the (a, b, d) space of Figures 2a and 2b.

A.3 Supplementary empirical results

Table 5: Marital assortativeness: comparing cohort born 1970-75 with earlier cohorts

	Submarkets			Global indices	
	C&PG vs SC	PG vs C	C vs SC	C&PG vs others	5 educ levels
	(1)	(2)	(3)	(4)	(5)
	Panel A: Cohorts 1970s vs 1930s				
Odds ratio	2.681	-1.965	1.912	-6.252	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
L1 ratio (high ed)	0.071	-0.290	0.163	-1.889	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
L2 ratio (low ed)	0.299	0.063	0.092	0.268	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
Weighted L ratio	0.116	0.043	0.133	0.327	0.383
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]
Normalized trace	0.102	0.011	0.084	-0.083	-0.028
	(.00) [.00]	(.23) [.14]	(.00) [.00]	(.00) [.00]	(.00) [.00]
χ^2	0.097	-0.010	0.080	0.060	
	(.00) [.00]	(.13) [.13]	(.00) [.00]	(.00) [.00]	
Minimum distance	0.022	-0.300	0.044	-0.080	
	(.20) [.10]	(.00) [.00]	(.01) [.00]	(.00) [.00]	
	Panel B: Cohorts 1970s vs 1940s				
Odds ratio	1.972	-0.852	1.597	-2.384	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
L1 ratio (high ed)	0.048	-0.075	0.109	-0.765	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
L2 ratio (low ed)	0.198	0.000	0.091	0.189	
	(.00) [.00]	(.98) [.98]	(.00) [.00]	(.00) [.00]	
Weighted L ratio	0.078	-0.014	0.100	0.195	0.256
	(.00) [.00]	(.32) [.20]	(.00) [.00]	(.00) [.00]	(.00) [.00]
Normalized trace	0.063	0.016	0.062	-0.037	0.005
	(.00) [.00]	(.01) [.00]	(.00) [.00]	(.00) [.00]	(.24) [.08]
χ^2	0.070	-0.013	0.064	0.028	
	(.00) [.00]	(.06) [.03]	(.00) [.00]	(.00) [.00]	
Minimum distance	0.003	-0.191	0.035	-0.059	
	(.93) [.77]	(.00) [.00]	(.01) [.00]	(.00) [.00]	
	Panel C: Cohorts 1970s vs 1960s				
Odds ratio	0.277	-0.170	0.236	0.279	
	(.48) [.16]	(.48) [.18]	(.53) [.20]	(.69) [.27]	
L1 ratio (high ed)	-0.054	-0.078	-0.042	-0.270	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
L2 ratio (low ed)	0.152	0.026	0.058	0.105	
	(.00) [.00]	(.00) [.00]	(.00) [.00]	(.00) [.00]	
Weighted L ratio	-0.039	0.013	0.014	0.087	0.130
	(.00) [.00]	(.49) [.17]	(.32) [.10]	(.00) [.00]	(.00) [.00]
Normalized trace	0.015	-0.012	0.003	-0.015	-0.002
	(.00) [.00]	(.05) [.01]	(.79) [.39]	(.00) [.00]	(.87) [.52]
χ^2	-0.001	-0.002	0.008	0.016	
	(.88) [.88]	(.87) [.65]	(.52) [.20]	(.01) [.00]	
Minimum distance	0.012	-0.042	0.026	0.068	
	(.49) [.20]	(.00) [.00]	(.05) [.02]	(.00) [.00]	

Notes: Education in 5 groups: post-graduate degrees (PG), 4-years college degrees (C), some college (SC), high school qualifications (HS), no qualifications (HSD). Column 1 refers to matching matrices between PG and C together and SC; column 2: PG and C; column 3: C and SC; column 4: PG and C together versus others; column 5: all 5 groups separate. For each index, top row shows estimates of the difference between the latest and earliest cohorts; the number in round brackets shows p-values for 2-sided significance testing adjusted for multiple hypothesis using the stepdown method jointly for the 30 measures on the panel (Romano and Wolf, 2005; Romano, Shaikh, and Wolf, 2008; Romano and Wolf, 2016); the number in square brackets shows stepdown p-values but for each market (column) separately. *Data source*: March extract of the US CPS, subsample of legally married and cohabiting individuals observed aged 35-44 in birth cohorts 1930-39, 1940-49, 1950-59, 1960-69 and 1970-75.