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Reputational Bargaining with External Resolution Opportunities

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Two parties negotiate in the presence of external resolution opportunities (*e.g.* court, arbitration, or war). The outcome of external resolution depends on the privately held justifiability/strength of their claims. A justified party issues an ultimatum for resolution whenever possible, but an unjustified party strategically bluffs with an ultimatum to establish a reputation for being justified. We show that the availability of external resolution opportunities can benefit or hurt an unjustified party in equilibrium. When the chances of being justified become negligible, agreement is immediate and efficient; and if the set of justifiable demands is rich, our solution modifies the Nash–Rubinstein bargaining solution of Abreu and Gul ((2000), *Econometrica*, **68**, 85–117) in a simple way.

Key words: Reputational bargaining, Ultimatum, Conflict resolution, Arbitration, War

JEL codes: C78, C79, D74

1. INTRODUCTION

In many negotiations, involved parties can seek external resolution (*e.g.* court, arbitration, or war) if internal resolution fails. In addition, they hold private information that determines the outcome of external resolution. For example, two parties involved in a patent infringement dispute can proceed to an intellectual property court if settlement fails, and the court can determine whether the plaintiff is a victim or a patent troll. Teams and players in Major League Baseball and the National Hockey League have for decades used league-provided arbitration if they fail to reach contract agreement: A panel of judges picks one of the two sides' publicly announced demands based on their privately prepared arguments. A country can threaten to invade another

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country if peaceful negotiation fails, and the outcome of the invasion depends on the countries' private military strength and devotion to the dispute.[1]

Threatening external resolution is frequently leveraged as a strategic posture in the form of an ultimatum for internal resolution.[2] The ability to make such a threat may vary by situation and location. For example, several large jurisdictions (*e.g.* California, Illinois, and Texas) have rules that explicitly bar attorneys from threatening disciplinary or criminal action to gain the upper hand in settlement talks. Some states (*e.g.* New York) only prohibit threatening criminal action, and other states (*e.g.* Michigan) have not enacted any rules in this area. A stated main motivation for prohibiting such a threat is its perceived unfair benefit to the aggressor when it is used (Shavell 2019; Hunter 2020). However, what is not considered is that not invoking external resolution may be a sign of weakness for the aggressor, and this may affect the frequency of internal resolution and the division of surplus. Hence, it is unclear who benefits from external resolution opportunities when we take equilibrium effects into account.

The prevalence, importance, and complexity of negotiation with external resolution opportunities warrant detailed investigation in a unified equilibrium framework. We focus on two questions. First, what are the effects of the presence of external resolution opportunities on involved parties' strategic behaviour and bargaining power? Second, which features of the negotiation process are essential determinants of the outcome as private information vanishes?

To address these questions, we incorporate external resolution opportunities in the continuous-time war-of-attrition bargaining model of Abreu and Gul (2000) (AG henceforth), which only involves internal resolution. In our model, Players 1 (he) and 2 (she) negotiate to divide a unit pie. Privately, each player is either justified or unjustified in their demand. A justified player demands a fixed share of the pie and never gives in to an offer smaller than their demand (which corresponds to the behavioural type in AG), and an unjustified player can demand any share and give in to any demand (which corresponds to the rational type in AG). Players announce their demands sequentially at the beginning of the game. Afterward, each player can (1) continue the negotiation by holding on to the announced demand or (2) end the negotiation by either giving in to the opposing demand (internal resolution) or challenging the opponent before external resolution with an ultimatum for internal resolution. Challenge opportunities arrive randomly for justified players, and unjustified players can always bluff.[3] Upon being challenged, the opponent must respond by either giving in to the challenger's demand (internal resolution) or seeing the challenge (external resolution).

The outcome of the external resolution depends on the justifiability of players' claims, which renders our model one of the first to study reputational bargaining with *interdependent values*. In court, the outcome can be determined by a judge who observes the justifiability of players' claims. In war, the outcome depends on countries' devotion and strength. If an auditor or mediator who reveals information is invoked, the outcome is the equilibrium payoff in the continuation game after players' claims are verified (as in Fanning (2021)).

In the model in which neither player has external resolution opportunities (the AG model), the unique equilibrium bargaining and reputation dynamics are parsimoniously characterised

1. Supplementary Material, Appendix A describes in more detail more applications in the realm we consider.

2. The threat of external resolution mainly serves as a strategic posture, because many disputes are resolved before external resolution is invoked. For example, 98% of criminal cases and 97% of civil lawsuits are resolved before trial, and 80% of financial arbitration cases and 95% of NHL salary arbitration cases are settled before their scheduled hearings (Gramlich 2019; Financial Industry Regulatory Authority 2020; National Hockey League Players' Association 2020). War is also arguably infrequent and often actively avoided.

3. We also consider the case in which bluffing is available randomly for unjustified players and demonstrate that our main results continue to hold (Supplementary Material, Appendix C.1).

as follows. After players announce their demands, at most one player concedes with a positive probability at time zero. Afterward, both players concede at constant hazard rates, and their reputations—the opponent's beliefs about a player's being justified—increase exponentially at the respective constant concession rates until both reputations reach one at the same time, at which point no unjustified player is left in the game and justified players continue to hold on to their demands.

We start our analysis with the case in which only one player—Player 1—has challenge opportunities.[4] This case is a building block for the setting in which both players have challenge opportunities, and most of the new economic forces from challenge opportunities on behaviour, reputation, and outcome are present and transparent in this case. We start with the setting in which each player has a single justified demand. In the unique equilibrium, as in the AG equilibrium, at most one player concedes with a positive probability at time zero, both players' concession rates are the same constant rates as in AG, and both players' reputations increase to one at the same time. In addition, an unjustified Player 1 challenges with a positive and increasing hazard rate as long as Player 2's reputation is not too high, and does not challenge at all after Player 2's reputation increases past a threshold (Theorem 1). Hence, in equilibrium, there is a challenge phase followed by a no-challenge phase.

The main methodological hurdle is the non-applicability of AG's solution method to our setting, which involves *interdependence* of players' payoffs and of their reputation-building processes: Player 1's strategy and reputation evolution depend on Player 2's reputation in each instance. To overcome this challenge, we introduce a new solution method based on *reputation coevolution diagram*, which is generally applicable in settings of interdependent payoffs. It encompasses AG, and is a central tool for our subsequent analysis of the baseline model and its extensions. We elaborate on our method in the subsection on related literature.

With the unique equilibrium characterised by the reputation coevolution diagram, we answer the two main questions stated above.

The first question is the equilibrium impacts of the introduction of challenge opportunities. Conceptually, the ability to challenge creates more possibilities for a player. However, not challenging when the opportunity is available reveals one's weakness, and that information could influence their bargaining power negatively. Namely, two forces in our model determine the speed and dynamics of reputation building. The first is reputation building by not conceding (not invoking internal resolution, as in AG): Persisting longer in the negotiation increases a player's reputation. The second, which is new in our model, is reputation gain or loss by not challenging (not invoking external resolution). On one hand, the presence of challenge opportunities can hurt Player 1 by slowing reputation building, when an unjustified Player 1 is expected to challenge at a lower rate than a justified Player 1. This is because not challenging is evidence against his being justified (bad-news). On the other hand, the presence of challenge opportunities can benefit Player 1 by speeding up reputation building and increasing Player 2's concession probability at the beginning of the game, when an unjustified Player 1 is expected to challenge at a higher rate than a justified Player 1 (good-news). What is the net equilibrium impact of challenge opportunities? Player 1's equilibrium payoff may be higher or lower with the presence of challenge opportunities. In particular, the presence of challenge opportunities may benefit an unjustified challenger only when he has an intermediate level of prior reputation; this holds even if the challenge opportunities arrive very frequently. Moreover, the challenger may never benefit

4. For example, in MLB and the NHL, essentially only players can elect to have salary arbitration hearings; in civil lawsuits, usually only one side has the incentive to sue the other side; in price negotiations, typically either the buyer or seller—but not both—waits for outside options; and in international conflicts, one side may consider aggression.

from challenge opportunities if the cost of challenge is high and/or the cost of response is low. Uncertainty about the beneficiary of the presence of challenge opportunities helps rationalise aforementioned disparate approaches to allowing legal threats (Shavell 2019; Hunter 2020).

Second, we find that when initial reputations approach zero (both players are rational with probability close to one), the equilibrium outcome depends on a minimal set of details of the setting. In this so-called limit case of rationality, the equilibrium outcome is efficient: One player yields to the opponent's demand at time zero with a probability approaching one (Proposition 1). The identity of the loser—the player who concedes with probability one at time zero—and the surplus division are determined by the discount rates, demands, and ultimatum opportunity arrival rate via a simple formula. The set of parameters for which Player 1 loses expands with the ultimatum opportunity arrival rate; hence, ultimatum opportunities always hurt Player 1 in the limit case of rationality. In the context of the court, being able to threaten with an ultimatum opportunity does not necessarily benefit Player 1, and, in fact, always hurts Player 1 in the limit. In the context of international conflicts, this result suggests that threatening with a war as external resolution may hinder a country's ability to receive concessions from its rival.

Moreover, in a rich demand space the equilibrium outcome is unique (Theorem 2), and the presence of ultimatum opportunities affects players' bargaining power in a remarkably simple way. As initial reputations approach zero, and as the set of justified demands gets larger and finer, the equilibrium outcome converges to a unique efficient division that only depends on discount rates and the ultimatum opportunity arrival rate (Proposition 2). In particular, Player 1's equilibrium payoff is the AG payoff if the ultimatum opportunity arrival rate is smaller than his discount rate, and is equal to what his AG payoff would be if his discount rate were replaced by the ultimatum opportunity arrival rate if the rate is larger than his discount rate. In the former case with slow arrival of ultimatum opportunities, players tend to compromise; in the latter case with fast arrival of ultimatum opportunities, Player 2 chooses the greediest demand to discipline Player 1.

An application of our model is the formation of a defense alliance between countries, which can be interpreted as a committed response to an ultimatum in the hope of deterring unjustified aggressors. We study the implications of joining a defense alliance on payoffs and conflict frequency. We show that Player 2 may benefit from the deterrence effect of a defense alliance when her reputation is low but will be hurt when her reputation is high. Moreover, although commitment deters aggression (from unjustified players in our model), it may increase conflict (from committing to respond to justified players). The overall ambiguous effect is consistent with the division in the literature regarding whether a defense alliance deters or provokes conflict (Kenwick *et al.* 2015; Leeds and Johnson 2017; Morrow 2017).

After the literature review, the rest of the paper proceeds as follows. Section 2 describes the basic model with one-sided ultimatum opportunities, and Section 3 characterises its equilibrium. Section 4 discusses the determinants of bargaining outcome, Section 5 summarises extensions and concludes, and Section 6 collects omitted proofs. Supplementary Material, Appendices provide omitted details.

1.1. *Relation to the literature*

Our paper builds on the seminal work of Abreu and Gul (2000), who were the first to study two-sided reputational bargaining as a concession game.[5] They show the convergence of the

5. Chatterjee and Samuelson (1987) study a discrete-time concession game with two-sided incomplete information (war-of-attrition). Myerson (1991) introduces one-sided reputational bargaining. Subsequent contributions to

equilibrium outcomes of discrete-time bargaining games with incomplete information to the unique equilibrium of a continuous-time war-of-attrition model.

We build on their war-of-attrition model by adding the opportunity for players to challenge and seek external resolution. When the exogenous arrival rate of ultimatum opportunities to the justified type is zero, our model is equivalent to AG's model. When this arrival rate is strictly positive, a new possibility of external resolution arises. Compared with AG, our model requires new techniques and allows us to study a wider range of applications. Specifically, (1) the addition of ultimatum opportunities results in richer yet tractable strategic behaviour and reputation dynamics, solved by new methods and aided by the introduction of reputation coevolution diagrams; (2) even though external resolution disfavours unjustified players, its availability may benefit them in equilibrium through reputation building; and (3) payoffs in the limit case of rationality and rich type spaces modify AG's payoffs in a simple way.

Our analysis differs from AG's in two main technical aspects. First, in our model players have a larger strategy space due to the additional challenge opportunities. A priori, players may have more or less incentive to wait to concede due to anticipating challenges. However, we show that in equilibrium, a player's payoff when challenged is equal to the payoff from conceding. Moreover, the equilibrium distribution of challenges is continuously strictly increasing up to a finite time, and halts afterward. These findings show that the equilibrium structure of our model is a tractable enrichment of AG's.

Second, more importantly, in AG's model players' equilibrium behaviour does not depend on their opponent's reputation, whereas in our model it inevitably does. AG develops a "forward-looking" method that first calculates the time it takes for each player's reputation to reach one in the absence of an initial concession to determine the winning player and then characterises the initial concession probability to ensure that players' reputations reach one at the same time. This method no longer applies to our model, because of the interdependence of the evolution of players' reputations. Instead, we develop a "backward-looking" method that characterises players' reputations jointly. The reputation coevolution curve, which depicts players' reputations as functions of each other's reputation, characterises the locus of players' reputations in any equilibrium of all games with all possible initial reputations after the start of the game. This locus divides the reputation plane into two regions that identify the winning player and the initial concession probability of the losing player.[6]

To the best of our knowledge, our paper is the first to study reputational bargaining with interdependent values.[7] In addition, our model is related to previous literature on bargaining with deadlines or outside option and conflict resolution. The ultimatum in our model can be seen as invoking an immediate deadline. Fanning (2016) studies reputational bargaining with exogenous deadlines, and obtains a monotonic hazard rate of dispute resolution when the deadline distribution is tightly compressed in a time interval. In our model, we assume that the arrival rate of ultimatum opportunities for the justified type is constant, yet we obtain a piecewise monotonic

reputational bargaining include Kambe (1999), Abreu and Pearce (2007), Wolitzky (2011, 2012), Atakan and Ekmekci (2013), Abreu *et al.* (2015), and Sanktjohanser (2023). See Fanning and Wolitzky (2020) for a comprehensive survey.

6. Kreps and Wilson (1982) and Fudenberg and Kreps (1987) use a similar representation of the state space with two players' reputations, but they do not use the reputation coevolution curve to derive the probability of initial concession or pin down additional strategy dynamics.

7. Reputational bargaining with interdependent values naturally arises in settings of bargaining under different information structures, type-dependent outside options, and mediation or arbitration in which the mediator or arbitrator suggests or enforces an outcome that depends on bargainers' types. See Pei (2020) for reputation effects under interdependent values.

rate of dispute resolution in the middle of the negotiation due to the endogeneity of ultimatum usage rates by strategic players. In addition, we obtain discontinuity in the hazard rate of resolution due to the endogeneity of payoffs when an ultimatum is issued. Relatedly, Fanning (2021, 2023) studies a reputational bargaining model in which a mediator makes nonbinding recommendations at the beginning of negotiation. In our model, our third party resembles an arbitrator who imposes a binding resolution when consulted during the negotiation.

Another interpretation of the ultimatum is an endogenously evolving outside option. A player can use an ultimatum to let a third party divide the surplus. Compte and Jehiel (2002) study exogenous outside options that generate a value strictly higher than concession, and show that these high-value outside options cancel out reputation effects. Atakan and Ekmekci (2014) study reputational bargaining in a market setting with many buyers and sellers. In their model, the market serves as the endogenous outside option, and they show that even in the limit case of rationality inefficiency may arise. We obtain a similar inefficiency result when both players can challenge frequently and when the probability of being justified is small. In addition, the models of Özyurt (2014, 2015) share the similarity whereby the value of the outside option depends on players' evolving reputations, but the papers' motivations and modelling choices differ otherwise. There is a further related literature on the exogenous arrival of outside options in bargaining with one-sided incomplete information. In Hwang and Li (2017) and Hwang (2018), not taking an outside option opens up the possibility of nonincreasing reputations and equilibrium multiplicity. Lee and Liu (2013) study the role of incomplete information and outside options in bargaining, but between a long-run player and a sequence of short-run players.

The paper is also related to conflict bargaining and defense alliances in international relations (Fearon 1994; Sandroni and Urgun 2017, 2018). This literature studies situations in which players can end the bargaining process by confronting each other. However, in these models, not ending the bargaining process is more efficient, and the equilibrium dynamics are different from the war-of-attrition dynamics in our paper. Fearon (1994) demonstrates the importance of audience costs (*i.e.* waiting costs) in the bargaining outcome; our limit result shows that the bargaining outcome depends on bargaining costs in a simple way, but does not depend on court costs. Our application also sheds light on the deterrence versus provocation effect of a defense alliance (Kenwick *et al.* 2015; Leeds and Johnson 2017; Morrow 2017).

2. MODEL

Players 1 (he) and 2 (she) decide how to split a unit pie. Each player is either (1) *justified and committed* in demanding a fixed share of the pie or (2) *unjustified and strategic* in demanding any fixed share.

We start by assuming that each player can be one single justified type: With probability z_1 Player 1 is justified in demanding $a_1 \in (0, 1)$, and with probability z_2 Player 2 is justified in demanding $a_2 > 1 - a_1$. Let $D := a_1 + a_2 - 1$ denote the amount of disagreement between the two players.

Time is continuous and the horizon is infinite. At time zero, Player 1 announces his demand first, and upon observing Player 1's announcement, Player 2 accepts it or announces her demand.[8] At each instant, each player can either concede to their opponent or not concede. We assume that a justified player never concedes. When an unjustified Player i concedes to Player j, Player i gets a payoff of $1 - a_j$ and Player j gets a payoff of a_j. In addition, we start by assuming

8. Because for now there is only a single justifiable type, the initial demand announcement stage is redundant. When we allow for multiple types, we add the demand announcement stage.

TABLE 1
Outcome in the external resolution

	Unjustified defender	Justified defender
Unjustified challenger	$1-a_2+w_1D, 1-a_1+(1-w_1)D$	$1-a_2, \cdot$
Justified challenger	$\cdot, 1-a_1$	$\cdot, \cdot$

Note: $\cdot$ indicates that the payoff is irrelevant for the strategic consideration of an unjustified player.

a one-sided challenge model: Player 1 has opportunities to challenge Player 2 with an ultimatum. The game ends upon a concession if a player concedes, or after the challenge stage if a player challenges.

It costs c_1D for Player 1 to challenge. A justified Player 1's challenge opportunities arrive according to a Poisson process with rate $\gamma_1 \in [0, \infty)$, and he challenges whenever such an opportunity arrives. An unjustified Player 1 can challenge at any time, so he can time his challenge strategically and *bluff* with an ultimatum. Player 2 can respond to a challenge either by yielding to the challenge or by seeing it. A justified Player 2 always sees a challenge, and an unjustified Player 2 chooses between the two actions. If Player 2 yields, she gets $1-a_1$ and Player 1 gets a_1. It costs k_2D for Player 2 to see a challenge, and in this case the division of the pie is determined by external resolution.

We start with the extreme case in which the external resolution always favours a justified player against an unjustified player. If an unjustified player meets a justified player, the unjustified Player i receives $1-a_j$. If two unjustified players meet, either the challenging Player 1 is favoured and gets a_1 (with probability w_1), or is disfavoured and gets $1-a_2$ (with probability $1-w_1$). Therefore, his expected share is $1-a_2+w_1D$ and defending Player 2's expected share is $1-a_1+(1-w_1)D$. Players' payoffs are linear in the share of the surplus they receive, so we could equivalently interpret that the third party decides on a deterministic compromise division that gives each player their respective expected share. We do not specify the outcome for justified players, since this does not play any role in the strategic decisions of unjustified players. Table 1 summarises the outcome of the external resolution considered in the benchmark model.

In the benchmark model, we assume that if Player 2 is expected to see a challenge, then Player 1 prefers conceding to challenging: $w_1 < c_1 < 1$; and that Player 2 prefers seeing a challenge from an unjustified Player 1 to yielding to it: $0 < k_2 < 1-w_1$. If $w_1 = 0$—*i.e.* the external resolution never favours an unjustified plaintiff—we are simply assuming c_1 and k_2 to be strictly between 0 and 1.[9]

In summary, a bargaining game $B = (a_1, a_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ with ultimatum opportunities for one player and single demand types for both players is described by players' justified demands a_1 and a_2, prior probabilities z_1 and z_2 of being justified, discount rates r_1 and r_2, challenge opportunity arrival rate γ_1 for a justified Player 1, challenge cost c_1 and seeing cost k_2 as proportions of the conflicting difference, and an unjustified Player 1's winning probability w_1 against an unjustified opponent. Supplementary Material, Appendix A provides several applications that can be thought of as negotiation with external resolution opportunities.

2.1. *Formal description of strategies and payoffs*

Since only unjustified players can choose their strategies, we drop the qualifier "unjustified" or "strategic" whenever no confusion can arise. An unjustified Player 1's strategy is described

9. In Supplementary Material, Appendix C.3, we explore alternative external resolution mechanisms.

by $\Sigma_1=(F_1,G_1)$, where F_1 and G_1, the probabilities of conceding and challenging by time (including) t, respectively, are right-continuous and increasing functions with $F_1(t)+G_1(t)\le 1$ for every $t\ge 0$. A strategic Player 2's strategy is described by $\Sigma_2=(F_2,q_2)$, where F_2, the probability of conceding by time t, is a right-continuous and increasing function with $F_2(t)\le 1$ for every $t\ge 0$, and $q_2(t)\in[0,1]$, her probability of yielding to a challenge at time t, is a measurable function. Each strategy profile induces a distribution over action profiles, which we refer to as *equilibrium play*.

A strategic Player 1's (time zero) expected utility from conceding at time t is[10]

$$U_1(t,\Sigma_2)=(1-z_2)\int_0^t a_1e^{-r_1s}dF_2(s)+[1-(1-z_2)F_2(t)]e^{-r_1t}(1-a_2)$$
$$+(1-z_2)[F_2(t)-F_2(t^-)]\frac{a_1+1-a_2}{2}, \quad (1)$$

where $F_2(t^-):=\lim_{s\uparrow t}F_2(s)$. His expected utility from challenging at time t is[11]

$$V_1(t,\Sigma_2)=(1-z_2)\int_0^t a_1e^{-r_1s}dF_2(s)+[1-(1-z_2)F_2(t)]e^{-r_1t}(1-a_2-c_1D)$$
$$+(1-z_2)[1-F_2(t)]e^{-r_1t}[(1-q_2(t))w_1+q_2(t)]D.$$

His expected utility from strategy Σ_1 is

$$u_1(\Sigma_1,\Sigma_2)=\int_0^\infty U_1(s,\Sigma_2)dF_1(s)+\int_0^\infty V_1(s,\Sigma_2)dG_1(s).$$

A strategic Player 2's expected utility from conceding at time t and yielding according to $q_2(\cdot)$ when facing a challenge is

$$U_2(t,q_2(\cdot),\Sigma_1)=(1-z_1)\int_0^t a_2e^{-r_2s}dF_1(s)+z_1\int_0^t[1-a_1-(1-q_2(s))k_2D]e^{-r_2s}\gamma_1e^{-\gamma_1s}ds$$
$$+(1-z_1)\int_0^t\{1-a_1+[1-q_2(s)][1-w_1-k_2]D\}e^{-r_2s}dG_1(s)$$
$$+e^{-r_2t}(1-a_1)[1-(1-z_1)F_1(t)-(1-z_1)G_1(t^-)-z_1\left(1-e^{-\gamma_1t}\right)]$$
$$+e^{-r_2t}(1-z_1)[F_1(t)-F_1(t^-)]\frac{a_2+1-a_1}{2}, \quad (2)$$

where $F_1(t^-):=\lim_{s\uparrow t}F_1(s)$. Her expected utility from strategy Σ_2 is

$$u_2(\Sigma_2,\Sigma_1)=\int_0^\infty U_2(s,q_2,\Sigma_1)dF_2(s).$$

We study this game's Bayesian Nash equilibria. Because the game is dynamic, it is natural to define public beliefs about players' types—*i.e. reputations*—throughout the game. We define

10. We assume an equal split when two players concede at the same time. This is inconsequential for our results, because simultaneous concession occurs with probability zero in equilibrium.

11. We assume that whenever concession and challenge occur simultaneously, the outcome is determined by the concession. This is an innocuous assumption, because simultaneous concession and challenge occur with probability 0 in equilibrium.

the reputation process $\mu_i(t)$ in the natural way, as the posterior belief that player i is justified conditional on the game's not ending by time t. Bayes' rule gives us this process explicitly as

$$\mu_1(t) = \frac{z_1\left[1-\int_0^t \gamma_1 e^{-\gamma_1 s} ds\right]}{z_1\left[1-\int_0^t \gamma_1 e^{-\gamma_1 s} ds\right] + (1-z_1)[1-F_1(t^-)-G_1(t^-)]},$$

and

$$\mu_2(t) = \frac{z_2}{z_2 + (1-z_2)\left[1-F_2(t^-)\right]}.$$

Finally, let $\nu_1(t)$ be Player 2's posterior belief that Player 1 is justified conditional on Player 1 challenging at time t. Namely, $\nu_1(t)=0$ at any $t \geq 0$ where G_1 has an atom, and at any $t \geq 0$ where G_1 is differentiable,

$$\nu_1(t) = \frac{\mu_1\gamma_1}{\mu_1\gamma_1 + [1-\mu_1(t)]\beta_1(t)}, \tag{3}$$

where

$$\beta_1(t) = \frac{G_1'(t)}{1-F_1(t^-)-G_1(t^-)}$$

is an unjustified Player 1's hazard rate of challenging.[12]

2.2. *Modelling choices and extensions*

We assume that challenge opportunities arrive stochastically. For example, the chance of invoking the court is not always available and is mostly private: It depends on the availability of the court, the availability of attorneys willing to take the case, and/or the availability of material evidence that supports a party's claim. Moreover, we model the arrival according to a Poisson process, which implies a constant arrival rate, but our analyses do not require the arrival process to be Poisson.

For expositional ease of equilibrium characterisation, we mainly study the "asymmetric" case, in which the unjustified player can challenge at any time and the justified player challenges only when the opportunity arrives. In Supplementary Material, Appendix C.1.1, we study the "symmetric" case, in which challenge opportunities arrive equally frictionally for justified and unjustified Player 1. We extend the equilibrium characterisation and demonstrate the generality of the key results established in the "asymmetric" benchmark model.

Also for expositional ease, we start with the perfect association of commitment behaviour (*i.e.* always challenging and always seeing a challenge) with justified players, who get a favourable outcome in the external resolution. For example, when the external resolution mechanism is an auditor or mediator who publicly reveals players' types, the perfect association between the commitment behaviour and being justified is natural. This is because when an auditor or mediator is called upon to reveal the commitment behaviour of players, there is a perfect association of being committed and being justified and that of being strategic and being unjustified in the following sense. When the auditor reveals a party to be committed and the other party

12. The function G_1 is differentiable almost everywhere, because it is right-continuous and monotone. Moreover, the posterior beliefs are well defined at the jump points of G_1, and hence they are well defined almost everywhere in both the G_1 measure and Lebesgue measure.

to be rational, in the continuation game the rational party concedes. When both parties are rational, the continuation payoffs are efficient and captured by the division share w_1 (as in Fanning (2021)).

In the current specification of external resolution, challenge is dominated by concession if an unjustified opponent always sees the challenge: $w_1 < c_1$. In Supplementary Material, Appendix C.3, we explore alternative external resolutions to showcase the versatility of our solution method. We consider the setting in which challenge dominates concession (*e.g.* if external resolution is random and/or if the challenge is costless) and the setting in which challenge neither dominates nor is dominated by concession (*e.g.* external resolution is noisy): $w_1 > c_1$. Moreover, our results continue to hold if Player 1 pays the court cost only when Player 2 sees the challenge.

We focus on the model with one-sided ultimatum opportunities, since it has many applications (*e.g.* patent infringement, debt collection, country aggression) and captures most of the economic channels under consideration. Supplementary Material, Appendix C.4 studies the model with two-sided ultimatum opportunities (which has a different set of applications—*e.g.* the division of financial assets in a dissolved firm) and highlights the similarities to (Supplementary Material, Appendix C.4.2) and differences from (Supplementary Material, Appendix C.4.3) the one-sided model.

We model the negotiation process directly as a concession game in the style of a war-of-attrition with the addition of ultimatum opportunities. We could alternatively model the negotiations in a *continuous-discrete-time* model in which a player can change his demand at any positive integer time, but can also concede to an outstanding demand (or challenge in our case) at any time $t \in [0, \infty)$. This formulation was introduced by Abreu and Pearce (2007) in a setting of repeated games with contracts and adopted by Abreu *et al.* (2015) in a bargaining context. In that formulation, without ultimatum opportunities, whenever a player makes a demand different from that of a commitment (justified) type she reveals her rationality, and there is a unique equilibrium continuation payoff vector, which coincides with the payoff vector from concession. With ultimatum opportunities, however, when Player 2 reveals rationality, there are multiple equilibria with different continuation payoffs. For example, there is an equilibrium in which Player 2 chooses a fixed demand, players concede to each other at constant hazard rates, Player 1 challenges at a constant rate, and Player 1's reputation remains constant. However, when Player 1 reveals rationality, there is a unique equilibrium continuation payoff vector, which coincides with the payoff vector from concession. In particular, all of the equilibria we identify in our model have an analogous equilibrium in the continuous-discrete-time bargaining model that yields identical behaviour.

3. EQUILIBRIUM

In this section, we solve and characterise equilibrium strategies and reputations.

3.1. *Equilibrium characterisation*

Theorem 1. *Consider $B = (a_1, a_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$, a bargaining game with one-sided ultimatum opportunities and single demand types. There exists an equilibrium. There exist finite times T and $T_1 \in [0, T)$ such that every equilibrium strategy profile $(\widehat{F}_1, \widehat{G}_1, \widehat{F}_2, \widehat{q}_2)$ satisfies the following properties.*

(1) $\widehat{F}_1$ and $\widehat{F}_2$ are strictly increasing in $(0, T)$ and constant for $t \geq T$;

(2) $\widehat{F}_1$ and $\widehat{F}_2$ are atomless in $(0, T]$ and at most one of the two has an atom at $t = 0$;

(3) *(a)* $\widehat{F}_1(T) + \widehat{G}_1(T_1) = 1$*;*

(b) $\widehat{F}_2(T) = 1$*;*

(4) *(a)* $\widehat{G}_1$ *is atomless in* $[0, T]$*, strictly increasing in* $[0, T_1]$*, and constant for* $t \geq T_1$*;*

(b) For almost every $t \in [0, T]$*,* $\widehat{q}_2(t) \in (0, 1)$ *if* $t \in [0, T_1]$ *and* $\widehat{q}_2(t) = 1$ *if* $t \in (T_1, T]$.

Moreover, $\widehat{F}_1$*,* $\widehat{F}_2$*, and* $\widehat{G}_1$ *are unique, and* $\widehat{q}_2$ *is unique almost everywhere for* $t \leq T$.

Property 1 states that there is a finite time $T > 0$ such that players concede to each other with a strictly positive probability in every subinterval of $(0, T]$, and never concede after time T. Property 2 states that the distributions of concession are atomless except at time zero, and there can be an atom in at most one of these distributions. Property 3a states that an unjustified Player 1 has either conceded before time T or challenged before time T_1, and Property 3b states that an unjustified Player 2 has conceded before time T. Properties 1, 2, and 3b coincide with the three properties in AG, and Property 3a modifies AG to characterise equilibrium challenge usage.

Property 4 extends AG's equilibrium characterisation when there are ultimatum opportunities. There are difficulties, however, due to players' larger strategy spaces: In addition to the timing of concession, Player 1 chooses the timing of challenge and Player 2 chooses how to respond to a potential challenge at each instant. A priori, players may have bigger or smaller incentives to concede due to the (anticipated) arrival of challenge opportunities at each instant. We first show that in every equilibrium, Player 2 does not benefit from challenges—*i.e.* at each instant she weakly prefers conceding to seeing a challenge. Second, we show that $\widehat{G}_1$ is atomless. These findings allow us to show that players' concession distributions are strictly increasing and atomless in an interval $(0, T_1)$.

Property 4a asserts that Player 1 challenges his opponent with an atomless distribution until some time $T_1 < T$, and never challenges afterward. Property 4b asserts that Player 2 responds to a challenge by both seeing the challenge and yielding to it with positive probabilities until time T_1, and yields to it afterward. Because this is a new property, let us provide an intuition for why this property must hold. Property 1 implies that at any time $t \in (0, T)$, Player i's continuation payoff at time t is equal to $1 - a_j$. If $\widehat{G}_1$ is constant in some interval, after observing a challenge in that time interval, Player 2's posterior belief that Player 1 is justified is one, and Player 2 optimally yields to any challenge. However, if Player 2's reputation is smaller than $\mu_2^* := 1 - c_1$, challenging gives Player 1 a payoff that strictly exceeds $1 - a_2$, which yields a contradiction. Similarly, if $\widehat{G}_1$ had an atom at some time t, then after observing a challenge at time t, Player 1's reputation would be 0 and Player 2 would optimally see the challenge. However, then Player 1 would receive a payoff strictly lower than $1 - a_2$, leading again to a contradiction. Furthermore, as we will argue in the next section, Player 2's reputation increases over time, and at some time $T_1 < T$ reaches μ_2^*. After this time, Player 1 never challenges. Finally, for time $t < T_1$, Player 1 is indifferent between conceding and challenging, and Player 2's reputation is smaller than μ_2^*. Therefore, $\widehat{q}_2(t) \in (0, 1)$ for time $t < T_1$.

We now use the four properties to derive the closed-form solutions of equilibrium strategies $\widehat{F}$ and $\widehat{G}$. In the next subsection, we first derive the equilibrium concession rates at time $t > 0$, Player 1's challenge rate, and Player 2's challenge response. We then derive reputation evolution based on these rates and construct a reputation coevolution diagram, which allows us to compute the probabilities of concession at time $t = 0$.

3.2. *Equilibrium strategies*

3.2.1. Challenge and response to challenge. Property 2 implies that Player 1 is indifferent between challenging and conceding at any time $t \in (0, T_1)$. At any such time t, $\mu_2(t)$ denotes

Player 2's reputation and $q_2(t)$ denotes the probability that Player 2 yields if a challenge comes at time t. Compared with conceding, the benefit of challenging comes from winning against an unjustified opponent who yields or sees, $[1-\mu_2(t)][q_2(t)+(1-q_2(t))w_1]D$, and the cost of challenging is $c_1 D$. Hence, we obtain that

$$q_2(\mu_2) := \frac{c_1 - w_1(1-\mu_2)}{1-\mu_2 - w_1(1-\mu_2)}. \tag{4}$$

The yielding probability is interior if $1 - c_1/w_1 < \mu_2 < \mu_2^* := 1 - c_1$. The lower bound is negative given the assumption that $c_1 > w_1$, and when μ_2 exceeds the upper bound $\mu_2^* = 1 - c_1$, the optimal choice is $q_2(\mu_2) = 1$. At any time $t \leq T_1$, Player 2 is indifferent between seeing and yielding to a challenge when Player 1's reputation conditional on challenging Player 2 is

$$\nu_1^* := 1 - \frac{k_2}{1-w_1} \Longleftarrow (1-\nu_1^*)(1-w_1)D - k_2 D = 0. \tag{5}$$

This implies, by Bayes' rule and equation (3), that Player 1's overall challenge rate seen as a function of Player 1's reputation is

$$\chi_1(t) = \frac{\mu_1(t)}{\nu_1^*}\gamma_1 \Longleftarrow \frac{\mu_1(t)\gamma_1}{\chi_1(t)} = \nu_1^*. \tag{6}$$

Equivalently, an unjustified Player 1's rate of bluffing with an ultimatum is

$$\beta_1(\mu_1) := \frac{1-\nu_1^*}{\nu_1^*}\frac{\mu_1}{1-\mu_1}\gamma_1 \Longleftarrow \frac{\mu_1\gamma_1}{\mu_1\gamma_1 + (1-\mu_1)\beta_1} = \nu_1^*. \tag{7}$$

To summarise, equation (4) holds almost everywhere for $t \leq T$, because actions after time T are off equilibrium path for an unjustified Player 2, and equation (6) holds almost everywhere for $t \leq T_1$ and $\beta_1(t) = 0$ almost everywhere for $t \in (T_1, T]$. As we will see, the reputations will increase over time in equilibrium, and unjustified Player 1's challenge rate β_1 will increase until Player 2's reputation increases to a threshold μ_2^*. Hence, the overall challenge rate χ_1 will increase until Player 2's reputation reaches μ_2^* and will then drop discontinuously and increase thereafter until Player 1's reputation reaches 1.

3.2.2. Concessions. Property 1 says that Player 1 concedes with a positive probability in every subinterval of $(0, T)$, so Player 1's continuation payoff at every time t is equal to $1 - a_2$ and he is indifferent between conceding at any time in $(0, T)$. Hence, Player 2 concedes at the constant rate λ_2 in the interval $(0, T)$ that sustains this indifference:

$$1 - a_2 = a_1\lambda_2 dt + e^{-r_1 dt}(1-a_2)(1-\lambda_2 dt) \Longrightarrow \lambda_2 = \frac{r_1(1-a_2)}{a_1+a_2-1},$$

as in AG; an unjustified Player 2 concedes at rate $\kappa_2 = \lambda_2/(1-\mu_2)$. An immediate implication is that Player 2's reputation conditional on negotiation continuing at time $t < T$, $\mu_2(t)$, is an increasing function.

Property 1 says that Player 2 concedes with a positive probability in every subinterval of $(0, T)$, as is the case for Player 1. However, from Player 2's perspective, in any time interval, Player 1 may concede or challenge. As Property 4b indicates, because Player 2 sees the challenge with an interior probability, her continuation payoff when she is challenged is equal to her payoff from conceding to Player 1. Hence, the indifference condition for Player 2 in yielding across all times $t \in (0, T)$ implies that the overall hazard rate of Player 1's conceding to Player 2 is

$\lambda_1 = r_2(1-a_1)/D$, as in AG. To summarise, each Player i, $i=1,2$, concedes at the overall rate of

$$\lambda_i := \frac{r_j(1-a_i)}{a_1+a_2-1} = \frac{r_j(1-a_i)}{D}. \tag{8}$$

3.3. *Equilibrium reputations*

3.3.1. Reputation evolution: bad-news and good-news effects. We now characterise the evolution of players' reputations. To do so, we use the concession rates and the challenge rate of Player 1 derived in the previous section. We start with Player 2's reputation building for $t \in (0,T]$. Player 1's reputation dynamics depend on both his concession rate and challenge rate. We start with the *no-challenge phase*, $t \in (T_1, T]$, and characterise the *challenge phase*, $t \in (0, T_1]$.

Note that Property 3 implies that $\mu_i(T)=1$ for $i=1,2$. Using this property and the reputation dynamics we derive, we characterise the *reputation coevolution curve*. This curve shows the locus of the reputation vectors at times $t>0$. The curve will determine the identity of the player who yields with a positive probability at time 0 and the magnitude of that atom. This will complete the characterisation of the unique equilibrium.

We use the Martingale property $\mu_i(t) = \mathbb{E}_t \mu_i(t+dt)$ to characterise players' reputation evolution in different phases, which can be succinctly summarised in the following lemma.

Lemma 1. *Player* 1*'s reputation evolution can be characterised as*

$$\dot{\mu}_1(t) := \frac{\mu_1'(t)}{\mu_1(t)} = \lambda_1 - \gamma_1 + \chi_1(t) \tag{9}$$

$$= \begin{cases} \lambda_1 - \left[1 - \dfrac{\mu_1(t)}{\nu_1^*}\right]\gamma_1 & \text{if } 0 < t \le T_1 \\ \lambda_1 - [1-\mu_1(t)]\,\gamma_1 & \text{if } T_1 < t \le T \end{cases} \tag{10}$$

and Player 2*'s reputation evolution is*

$$\dot{\mu}_2(t) := \frac{\mu_2'(t)}{\mu_2(t)} = \lambda_2. \tag{11}$$

Two forces shape the evolution of Player 1's reputation. First, no concession is good-news: With Player 1 conceding at rate λ_1, his reputation conditional on not having conceded increases exponentially at rate λ_1. The second force, which is new, comes from the equilibrium challenges. Observe that when $\gamma_1 = \beta_1(t) = 0$, equation (9) boils down to the exponential growth reputation dynamics in AG.

This second force can *decelerate* or *accelerate* reputation building. In the no-challenge phase, no challenge is bad-news: With a justified Player 1 challenging and an unjustified Player 1 not challenging at all, Player 1's reputation declines at rate $[1-\mu_1(t)]\gamma_1$. In the challenge phase, however, the unjustified Player 1 also challenges at a positive rate. Hence, the bad-news effect of no challenge is less severe in this phase than in the no-challenge phase. This is captured by the third term in equation (9). In fact, when $\beta_1(t) > \gamma_1$, Player 1's reputation building accelerates with no challenge, and no challenge becomes good-news. Player 1's reputation builds faster when $\mu_1(t) > \nu_1^*$—equivalently, $\beta_1(t) > \gamma_1$ and $\chi_1(t) > \gamma_1$—while Player 2's reputation is not too high, $\mu_2(t) < \mu_2^*$. This effect provides a benefit from the presence of ultimatum opportunities for an unjustified Player 1 who has an intermediate range of reputations. We characterise the range of initial reputations for ultimatum opportunities to be beneficial for an unjustified Player 1 in Section 4.2; this range may not exist in equilibrium. Decomposition of the bad-news and good-news effects clarifies the potential benefit of external resolution opportunities for an unjustified Player 1.

3.3.2. Reputation coevolution diagram. Both players' reputation dynamics in each phase follow the Bernoulli differential equation, which is one of the few cases of ordinary differential equations with closed-form solutions and includes the exponential growth of AG as the special case when ultimatum opportunities are absent. Hence, it is feasible to combine the reputation-building dynamics at different phases of the game to find the evolution of both players' reputations in equilibrium. To do so, we run the Bernoulli differential equations that describe players' reputation dynamics backward, starting from time T.

Recall that the finiteness of T in Property 3 of Theorem 1 implies that $\mu_1(T) = \mu_2(T) = 1$. Moreover, $\mu_2(T_1) = \mu_2^*$. Hence, $T - T_1$ can be found using Player 2's reputation dynamics given by equation (11). Then we can use Player 1's reputation dynamics in the no-challenge phase, equation (10), to find $\mu_1(T_1)$. Then we let T_1^* be the time it takes for Player 1 to build a reputation from z_1 to $\mu_1(T_1)$ using the dynamics in equation (10), and T_2^* the time it takes for Player 2 to build a reputation from z_2 to μ_2^* using the dynamics in equation (11). Finally, we let $T_2 := \min\{T_1^*, T_2^*\}$, and conclude that if $T_i^* > T_2$, Player i concedes at time 0 with a strictly positive probability.

Alternatively, we can trace a parametric *reputation coevolution curve* $(\mu_1(t), \mu_2(t))$ in the belief plane, which represents the locus of players' reputations for any initial reputations at any time $t > 0$. Because both reputations are characterised analytically, we can represent the graph of the coevolution curve as $\widetilde{\mu}_1(\mu_2)$ for $\mu_2 \in (0, 1]$, or equivalently, its inverse $\widetilde{\mu}_2(\mu_1)$ for $\mu_1 \in (\max\{0, \phi_1^* \nu_1^*\}, 1]$, where $\phi_1^* := 1 - \lambda_1/\gamma_1$. The coevolution curve is characterised by

$$\widetilde{\mu}_1(\mu_2) = \begin{cases} \dfrac{\lambda_1 - \gamma_1}{\lambda_1 (\mu_2)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} - \gamma_1} & \text{if } \mu_2^* < \mu_2 \leq 1, \\[2ex] \dfrac{\lambda_1 - \gamma_1}{\lambda_1 (\mu_2)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} + \left(\frac{\gamma_1}{\nu_1^*} - \gamma_1\right)\left(\frac{\mu_2}{\mu_2^*}\right)^{\frac{\gamma_1 - \lambda_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}} & \text{if } 0 < \mu_2 \leq \mu_2^*, \end{cases}$$

when $\gamma_1 \neq \lambda_1$. When $\gamma_1 = \lambda_1$, this curve is obtained directly from $\mu_1(t)$ and $\mu_2(t)$ or by applying L'Hôpital's rule to the above formula, and is explicitly given in Supplementary Material, Appendix B.1.3. We can obtain the reputation $\mu_1^N = \widetilde{\mu}_1(\mu_2^*)$ of Player 1 when Player 2's reputation is μ_2^*.

Figure 1 provides examples of the reputation coevolution curve in two cases. When $\gamma_1 \leq \lambda_1$, the curve tends toward $(0, 0)$ (Figure 1(a)), and when $\gamma_1 > \lambda_1$, since Player 1's reputation is decreasing for reputation lower than $\phi_1^* \nu_1^*$ in the challenge phase, the curve tends toward $(\phi_1^* \nu_1^*, 0)$ (Figure 1(b)). When (z_1, z_2) is on the coevolution curve, their reputations situate on the equilibrium path to $(1, 1)$, so neither player concedes at time 0 with a strictly positive probability. When (z_1, z_2) is to the left of the curve—that is, $\widetilde{\mu}_2(z_1) < z_2$—or equivalently, $\widetilde{\mu}_1(z_2) > z_1$, Player 1 will be the player who concedes with a positive probability at time 0. He must concede with a probability Q_1 such that the pair of his posterior reputation and Player 2's initial reputation z_2 exactly falls on the curve:

$$\frac{z_1}{z_1 + (1 - z_1)(1 - Q_1)} = \widetilde{\mu}_1(z_2) \Longrightarrow Q_1 = 1 - \frac{z_1}{1 - z_1} \Bigg/ \frac{\widetilde{\mu}_1(z_2)}{1 - \widetilde{\mu}_1(z_2)}. \tag{12}$$

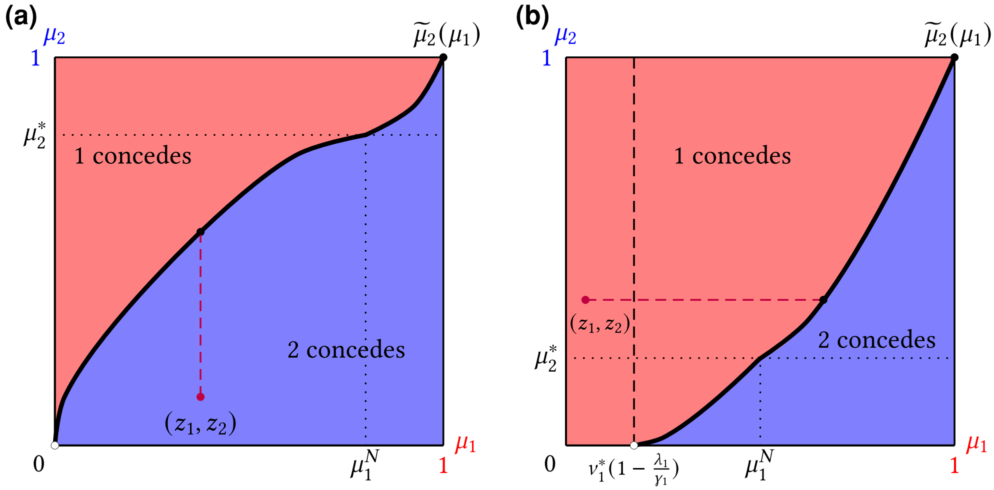

FIGURE 1

Reputation coevolution and initial concession in games with one-sided ultimatum opportunities. (a) $\gamma_1 \leq \lambda_1$, (b) $\gamma_1 > \lambda_1$

Notes: The solid line in each panel depicts the reputation coevolution curve $\widetilde{\mu}_2(\mu_1)$. Player 1 concedes with a positive probability at time 0 when (z_1, z_2) is strictly to the left of the curve, Player 2 concedes with a positive probability at time 0 when (z_1, z_2) is strictly to the right of the curve, and neither player concedes with a positive probability at time 0 when (z_1, z_2) is on the curve. The probability of initial concession ensures that the posterior reputation vector after initial concession lies on the curve. The reputations coevolve to (1, 1) according to the curve. When Player 2's reputation reaches μ_2^*, Player 1 stops challenging, and Player 1's reputation μ_1^N at the time is derived from the reputation coevolution curve.

When (z_1, z_2) is to the right of the reputation coevolution curve, Player 2 will be the one who concedes with a positive probability at time 0, which raises her reputation if she does not concede at time 0 to lie on the coevolution curve.

This completes our equilibrium characterisation. We summarise the resulting equilibrium strategies and beliefs explicitly in Supplementary Material, Appendix B.

4. IMPLICATIONS

4.1. *Rates of challenge and resolution*

Whereas distributions of challenging and dispute resolution depend on model primitives such as prior reputations and ultimatum opportunity arrival rates, some qualitative features of equilibrium hazard rates do not depend on the fine details of the model. For an unjustified Player 1, the equilibrium hazard rate of bluffing is increasing as t approaches T_1, and the rate of conceding is increasing to infinity as t approaches T.[13] Building on these rates, we can derive the overall hazard rates—that is, the aggregate rates for justified and unjustified players—of challenge and resolution.

Namely, an unjustified Player 1's equilibrium hazard rate of ultimatum usage increases between time 0 and time T_1 and drops to zero afterward, and an unjustified Player 1's equilibrium concession rate increases between time 0 and time T. The overall hazard rate of ultimatum usage increases between time 0 and time T_1, drops from $\frac{\mu_1^N}{\nu_1^*}\gamma_1$—which might be above or below

13. These rates are unique almost everywhere with respect to both the F_2 measure and Lebesgue measure.

γ_1—to a rate below γ_1, and increases to γ_1 between time T_1 and time T. The overall hazard rate of dispute resolution adds the concession rate $\lambda_1 + \lambda_2$ to the challenge rate before time T, and hence exhibits discontinuities at both time T_1 and time T.

A testable prediction of the model is that the hazard rate of resolution in negotiations, which we can observe in many settings, experiences local peaks and subsequent discontinuities in three instances: (1) the onset of negotiation, (2) the moment when an unjustified player stops challenging, and (3) the moment players stop conceding. The first peak arises when agreement is reached at the onset of the negotiation, the second peak arises when Player 2's reputation approaches the level beyond which Player 1 has no incentive to challenge, and the last peak arises when both players' reputations approach 1, beyond which neither player has an incentive to continue the negotiation. We predict some resolution in the middle of the negotiation, in addition to agreements at the onset of the game (predicted by Abreu and Gul (2000) and Fanning (2016)) and before the deadline (predicted by Fanning (2016), Simsek and Yildiz (2016), and Vasserman and Yildiz (2019)).[14] In an earlier version of the paper (Ekmekci and Zhang 2021), we provide evidence that suggests that there is also a spike in agreements after the beginning of the negotiation and before the deadline in MLB and NHL salary arbitration cases.

4.2. *Who benefits from access to external resolution?*

We now investigate the implications of external resolution on payoffs. This is important because some states (*e.g.* California, New York, Illinois, Texas) choose to restrict access to legal threat for the fear that it may be too powerful for a plaintiff (Shavell 2019; Hunter 2020). We show that when we consider the equilibrium effects of the access to external resolution on bargaining dynamics, access may hurt that player.

Whether a player benefits from access or not depends on the good-news and bad-news effects of ultimatum opportunities on reputation building described in Section 3.3. Recall that Player 1's reputation building slows in the no-challenge phase, because no challenge is a sign of weakness. In contrast, in the challenge phase, reputation building may be faster when an unjustified Player 1 bluffs at a rate higher than γ_1 (when the "no ultimatum is good news" effect dominates the "no ultimatum is bad-news" effect), which results in a benefit for an unjustified Player 1. Figure 2 illustrates whether an unjustified Player 1 benefits from the introduction of ultimatum opportunities.

Figure 2(a) illustrates the case in which Player 1 never benefits from the introduction of ultimatum opportunities. Figure 2(b) shows that when Player 1 challenges at a rate higher than γ_1, there may be an intermediate range of initial reputations of Player 1 in which he benefits from the introduction. We can show that this is always a connected interval bounded away from 0 and 1 when it exists.

This indeterminacy remains even when ultimatum opportunities arrive very frequently. This limit case of frictionless access to external resolution helps us clarify when and how an unjustified Player 1 benefits from access to external resolution. Since ultimatum opportunities arrive very frequently (essentially frictionlessly), the game ends quickly. Figure 3 illustrates reputation coevolution curves under large γ_1. As $\gamma_1 \to +\infty$, the reputation coevolution curve converges to the vertical line segment from $(\nu_1^*, 0)$ to (ν_1^*, μ_2^*), the horizontal line segment from (ν_1^*, μ_2^*) to $(1, \mu_2^*)$, and the vertical line segment from $(1, \mu_2^*)$ to $(1, 1)$ (but never reaches them for any finite γ_1), where $\nu_1^* = 1 - \frac{k_2}{1-w_1}$ and $\mu_2^* = 1 - c_1$. Hence, the equilibrium play can undergo a

14. We do not explicitly add a deadline to the model, but if we do, there will be a mass of deals near the deadline, and discontinuity in the hazard rates of challenge and resolution in the middle of the negotiation remains.

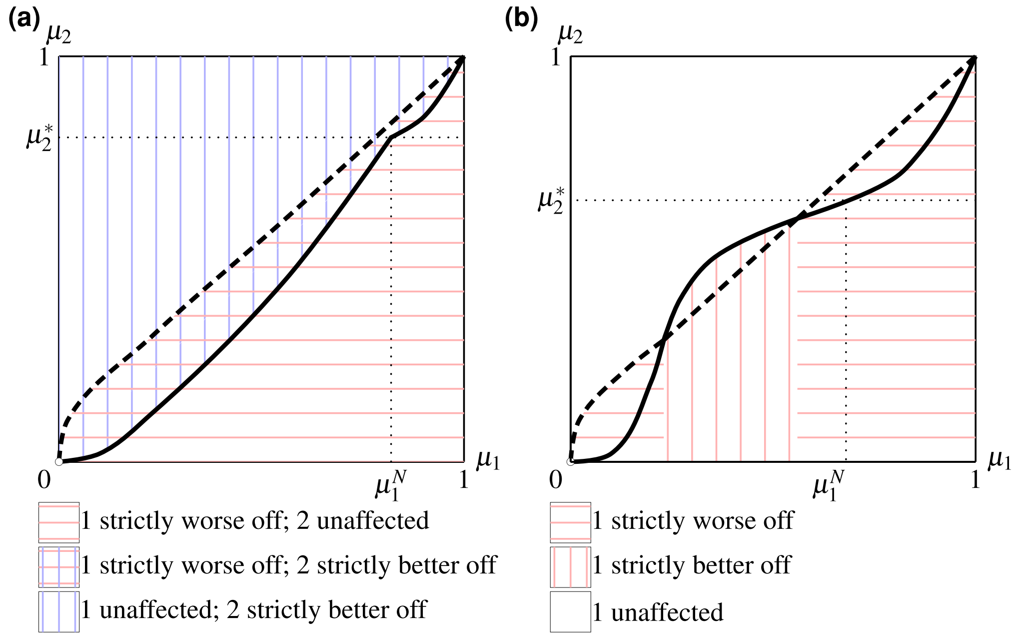

FIGURE 2

Does Player 1 benefit from the introduction of infrequent access to external resolution? (a) 1 is never strictly better off and (b) 1 may be strictly better off

Notes: The thick solid line depicts the reputation coevolution curve with $\gamma_1 > 0$, and the thick dashed line is the reputation coevolution curve with $\gamma_1 = 0$, as in AG. In (a), with the introduction of a challenge opportunity, Player 1 (resp., 2) is strictly worse off if the pair of initial reputations is in the region with horizontal lines, and is strictly better off if the pair of initial reputations is in the region with vertical lines. In (b), Player 2's change is not illustrated but can be analogously derived.

few reputation phases after an initial concession: Player 2's quickly increases (as it tends toward μ_2^*), Player 1's quickly increases (as it tends to μ_1^N), and then Player 2's quickly increases (from μ_2^* to 1).

Intuitively, the introduction of frequent ultimatum opportunities may benefit only Player 1 of intermediate reputation for the following reasons. For low initial reputations of Player 1 ($z_1 < \nu_1^*(1 - \frac{\lambda_1}{\gamma_1}) \approx \nu_1^*$), the introduction of frequent ultimatum opportunities helps quickly resolve uncertainty about players' justifiability, and the bad-news effect absolutely dominates when the strategy of persisting is dwarfed by justified players' quick challenges. For high initial reputations of Player 1 ($z_1 > (\mu_2^*)^{\lambda_1/\lambda_2}$), the introduction of frequent ultimatum opportunities renders the strategy of persisting at a finite rate ineffective, especially given that an unjustified Player 1 does not bluff. Only when Player 1 challenges at a rate higher than γ_1 does the good-news effect dominate and absolutely dominate (the speed of reputation building tends to infinity) as $\gamma_1 \to +\infty$. Whether Player 1 can benefit from the introduction of frequent ultimatum opportunities in equilibrium depends on the comparison between ν_1^* and $(\mu_2^*)^{\lambda_1/\lambda_2}$. When $\nu_1^* \geq (\mu_2^*)^{\lambda_1/\lambda_2}$, Player 1 never strictly benefits (Figure 3(a)). When $\nu_1^* < (\mu_2^*)^{\lambda_1/\lambda_2}$, for the intermediate range $(\nu_1^*, (\mu_2^*)^{\lambda_1/\lambda_2})$ of initial reputation of Player 1 (and when Player 2's reputation is below μ_2^*), an unjustified Player 1 strictly benefits from the introduction of frequent ultimatum opportunities (Figure 3(b)).

In the next subsection, we will analyse the case when players' probabilities of being justified are small and show that Player 1 is weakly worse off from access to external resolution opportunities, and increasingly worse off from more frequent access to these opportunities when their reputations are small.

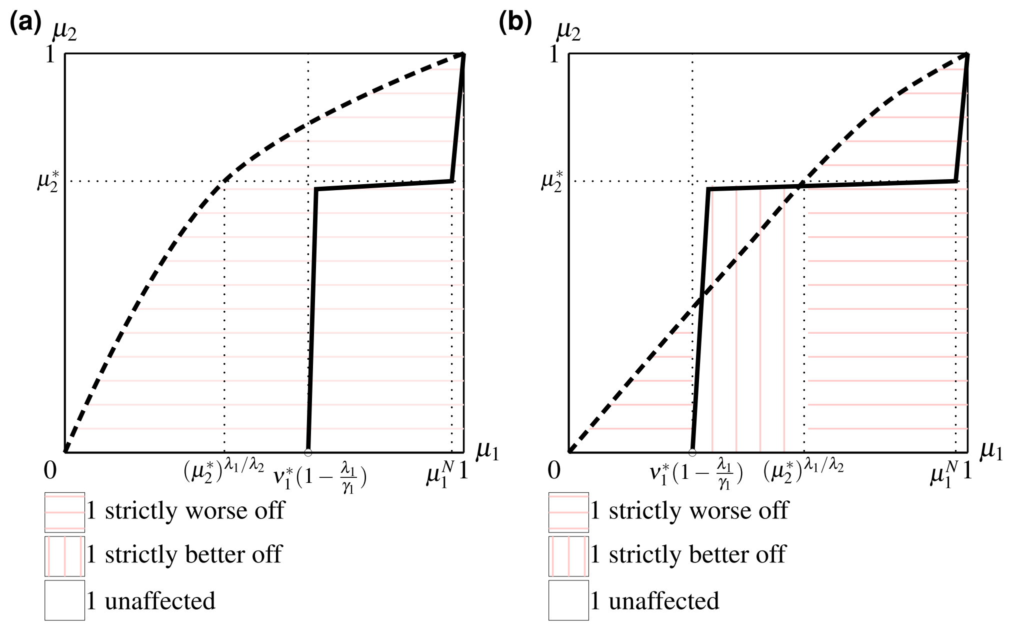

FIGURE 3

Does Player 1 benefit from the introduction of frequent access to external resolution? (a) 1 is never strictly better off and (b) 1 may be strictly better off

4.3. *Who benefits in the limit case of rationality?*

We now investigate the *limit case of rationality*, in which the prior probability that each player is justified is small. This case captures situations in which being justified is a rare event and an ultimatum is prominently used for strategic posturing.

4.3.1. Single-type space. We start with the case in which each Player i has a single justifiable demand a_i; we will allow the rational players to choose the justifiable in the next subsection. Generically (when $\lambda_1 \neq \gamma_1 + \lambda_2$, to be precise), players divide the surplus efficiently, with one player immediately conceding at time zero in equilibrium.

Proposition 1. *Let* $\{B^n\}_n$ *be a sequence of games in which for each* $n \in \mathbb{N}$, $B^n = (a_1, a_2, z_1^n, z_2^n, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ *is a bargaining game with one-sided ultimatum opportunities and single demand types. If* $\lim_{n\to\infty} z_1^n = \lim_{n\to\infty} z_2^n = 0$, *and* u_i^n *is the equilibrium payoff for player* i *in the game* B^n, *then*

$$\left(\lim_{n\to\infty} u_1^n, \lim_{n\to\infty} u_2^n\right) = \begin{cases} (1-a_2, a_2) & \text{if } \lambda_1 < \gamma_1, \text{ or} \\ & \text{if } \gamma_1 \leq \lambda_1 < \gamma_1 + \lambda_2 \quad \text{and} \quad \lim_{n\to\infty} z_1^n/z_2^n \in (0,\infty), \\ (a_1, 1-a_1) & \text{if } \lambda_1 > \gamma_1 + \lambda_2 \quad \text{and} \quad \lim_{n\to\infty} z_1^n/z_2^n \in (0,\infty). \end{cases}$$

If $\lambda_1 < \gamma_1$, the reputation coevolution curve approaches the x-axis at belief $\phi_1^* \nu_1^*$ (Figure 1(b)). Hence, for small z_1 and z_2, Player 1 concedes at time 0 with a large probability such that conditional on no concession, Player 1's reputation jumps above $\phi_1^* \nu_1^*$; we can verify this from equation (12).

If $\lambda_1 \geq \gamma_1$, the reputation coevolution curve approaches the x-axis at belief 0. In this case, when the prior probability of being justified goes to zero on the same order for the two players, agreement is efficient, is on the terms of Player 1 if $\lambda_1 - \gamma_1 > \lambda_2$, and is on the terms of Player 2 if $\lambda_1 - \gamma_1 < \lambda_2$. To see this, note that the derivative of the reputation coevolution curve, $\widetilde{\mu}_2'(\mu_1)$, as μ_2 goes to 0, tends to ∞ if $\lambda_1 - \gamma_1 > \lambda_2$ and tends to 0 if $\lambda_1 - \gamma_1 < \lambda_2$. Hence, as z_1 and z_2 go to 0 on the same order, Player 2 in the former case and Player 1 in the latter case concede at time 0 with a probability that approaches 1.

Note that limit payoffs are independent of the details of the arbitration; the costs of challenging and seeing the challenge as proportions of the disagreement, c_1 and k_2; and the probability w_1 of winning the challenge. Discount rates r_1 and r_2 and the ultimatum opportunity arrival rate γ_1 do not affect efficiency, although they determine who is the winner (the player who is conceded to immediately) and the loser (the player who concedes immediately). In particular, the higher the ultimatum opportunity arrival rate γ_1, the more likely Player 1 loses. Hence, unlike the general case in which the ultimatum opportunity may benefit or harm an unjustified Player 1, in the limit case of rationality, the ultimatum opportunity is always detrimental to an unjustified Player 1.

The intuition for this "independence from the details of external resolution" finding can be gained from the reputation dynamics. When z_1 and z_2 are small, negotiation may last for a long time—*i.e.* T is long. Moreover, reputation building for Player 1 takes the most time when $\mu_1(t)$ is small. Hence, Player 1's reputation increases approximately exponentially, and at the rate $\lambda_1 - \gamma_1$. In other words, it is as if the bad-news effect of not challenging slows the rate of reputation building exactly by γ_1. In light of our discussion in Section 3, this result shows that the good-news effect of challenging disappears and the bad-news effect persists for Player 1 in the limit case of rationality.

Finally, the player who builds reputation at the higher rate is the "winner"—*i.e.* their opponent concedes at time zero with a positive probability. Because reputations grow exponentially (approximately for Player 1), the initial concession probability converges to 1 as z_1 and z_2 approach 0 on the same order. This final part of our analysis is similar to that of Abreu and Gul (2000) and Kambe (1999).

4.3.2. Rich type space. We investigate the limit case of rationality when the set of available demand types for each player is sufficiently rich. The purpose of the analysis is to investigate which types stand out as the ones that are mimicked most often.

We first solve the case in which there are multiple justifiable demands for both players. Player 1 announces his demand $a_1 \in A_1$ first, and upon observing Player 1's announcement, Player 2 either accepts the demand or rejects the demand and announces her own demand $a_2 \in A_2$. Assume A_1 and A_2 are finite; assume that Player i's maximal demand is incompatible with all demands of Player j: $\max A_i + \min A_j > 1$. The prior conditional probability distribution π_i of demands by a justified Player i, in which $\pi_i(a_i)$ specifies the conditional probability of demanding a_i by a justified player, is commonly known. The game then proceeds as in the previous case with one-sided ultimatum opportunities and single demand types for both players. Hence, a game with one-sided ultimatum opportunities and multiple demands is described by the bargaining game $B = (\pi_1, \pi_2, z_1, z_2, r_1, r_2, \gamma_1, c_1, k_2, w_1)$. In addition to choosing their subsequent challenge, concession, and response to challenges, unjustified players choose initial demands to mimic.

Note that we model the costs of challenging and seeing a challenge as proportional to the disagreement in players' demands. This is without loss in the single-type case. However, with multiple demand types, this assumes a special relationship: The costs of challenging and seeing a challenge are proportional to the claimed disagreement between the two players. Our results in this section, Theorem 2 and Proposition 2, do not rely on this specific assumption, as long as the cost of challenging is higher than Player 1's expected gain $w_1 D$ and the cost of responding is lower than Player 2's expected gain $(1 - w_1)D$.

Let $\sigma_1 \in \Delta(A_1)$ denote an unjustified Player 1's mimicking strategy at the beginning of the game, and $\sigma_2(\cdot \mid a_1)$ an unjustified Player 2's upon observing Player 1's announced demand a_1, where the argument can be either any $a_2 \in A_2$ or $\{0\}$, which indicates the acceptance of Player 1's demand a_1.

Theorem 2. *For any bargaining game with ultimatum opportunities for one player and multiple demand types for both players, all equilibria yield the same distribution over outcomes.*

The proof is similar to the proof in AG. The key property—that players' payoffs are monotonic in z_i—is preserved in the current setting, as we will show in the comparative statics exercises. In the proof, we will first consider the intermediate case in which there is only one justified type of Player 1 but there are several justified types of Player 2. In this case, a unique equilibrium exists. Then we consider the general case in which Player 1 first chooses which type $a_1 \in A_1$ to mimic, and seeing this, Player 2 responds with a type $a_2 \in A_2$ to mimic. In this case, we show that the distribution of equilibrium outcomes is unique.

Note that the equilibrium outcome does depend on the order of the move. If Player 2 announces the demand before Player 1, then the distribution of equilibrium outcomes is still unique but potentially different from that when Player 1 announces first. However, as we will show, these orders will be irrelevant in the limit case of rationality and rich demand space.

For $K \in \mathbb{Z}_{>0}$, let $A^K := \{2/K, 3/K, \ldots, (K-1)/K\}$ be a set of demands. Each element of A^K corresponds to a commitment type whose demand coincides with that element. Suppose that $\pi_i \in \Delta(A^K)$ with full support—*i.e.* the prior distribution of Player i's type conditional on Player i being justified has full support on A^K. Finally, let z_i^n be the probability that Player i is a justified type. Hence, $z_i^n \pi_i(k/K)$ is the probability that Player i is a justified type who demands k/K, for $k = 2, \ldots, K-1$.

In what follows, we fix K and analyse any sequence of equilibrium outcomes of bargaining games in which the probabilities of each player being justified go to zero on the same order for the two players.

Proposition 2. *Let $\{B^n\}_n$ be a sequence of games in which for each $n \in \mathbb{N}$, $B^n = (\pi_1, \pi_2, z_1^n, z_2^n, r_1, r_2, \gamma_1, c_1, k_2, w_1)$ is a bargaining game with one-sided ultimatum opportunities and rich type spaces. If $\lim_{n\to\infty} z_1^n = \lim_{n\to\infty} z_2^n = 0$, $\lim_{n\to\infty} z_1^n / z_2^n \in (0, \infty)$, and u_i^n is the equilibrium payoff for Player i in the nth game of the sequence,*

$$\liminf u_1^n > \frac{r_2}{\max\{r_1, \gamma_1\} + r_2} - 1/K \quad \textit{and} \quad \liminf u_2^n > \frac{\max\{r_1, \gamma_1\}}{\max\{r_1, \gamma_1\} + r_2} - 1/K.$$

Remark 1. Proposition 2 implies that $\limsup u_i^n \leq 1 - \liminf u_{-i}^n$, because the size of the pie is 1. Therefore, as K grows without bound, Player 1's limit equilibrium payoff converges to $\frac{r_2}{\max\{r_1,\gamma_1\}+r_2}$ and Player 2's limit equilibrium payoff converges to $\frac{\max\{r_1,\gamma_1\}}{\max\{r_1,\gamma_1\}+r_2}$.

Proposition 2 illustrates how the bargaining power depends on the arrival of ultimatum opportunities in a remarkably simple way. The specific outcome of external resolution does not affect players' payoffs. Moreover, ultimatums have no impact if their arrival rate is smaller than the discount rate, and their arrival rate takes the role of the discount rate otherwise. Finally, when ultimatum opportunities are arbitrarily frequent—*i.e.* as $\gamma_1 \to \infty$—Player 2 guarantees herself the highest justifiable demand.

Proposition 1 shows that the limit equilibrium outcome when each side has a single type is (generically) efficient, *i.e.* agreement is immediate. Moreover, Player 1 wins if $\lambda_1 - \gamma_1 > \lambda_2$, and Player 2 wins if $\lambda_1 - \gamma_1 < \lambda_2$. In terms of the primitives of the model, Player 1 wins if

$$r_2(1-a_1) > r_1(1-a_2) + \gamma_1(a_1+a_2-1),$$

and Player 2 wins if the strict inequality sign is flipped. Note that in AG, the comparison is between $r_2(1-a_1)$ and $r_1(1-a_2)$—two terms that resemble the marginal costs of waiting that involve only demands and discount rates—to determine the winner. The comparison in our model is complicated by an additional term that involves the ultimatum opportunity arrival rate γ_1 and the amount of disagreement D. Addition of ultimatum opportunities cannot be thought of as simply a discount rate. Player i's problem is to maximise a_i subject to being the winner.

In the case of $\gamma_1 \leq r_1$, which includes $\gamma_1 = 0$ in AG as a special case, Player 1 can guarantee being the winner by choosing the demand $\max\{a_1 \in A^K \mid a_1 \leq \frac{r_2}{r_1+r_2}\}$. The result holds because the inequality above can be rearranged as

$$r_2(1-a_1) > (\gamma_1 - r_1)(a_1+a_2-1) + r_1 a_1 \iff r_2 - (r_1+r_2)a_1 > (\gamma_1 - r_1)(a_1+a_2-1).$$

Given the negative term on the right-hand side of the inequality, Player 1's Rubinstein-like demand guarantees his being the winner. Analogously, Player 2 is the winner if

$$r_1 - (r_1+r_2)a_2 > (-\gamma_1 - r_2)(a_1+a_2-1),$$

and she can guarantee being the winner by demanding $\max\{a_2 \in A^K \mid a_2 \leq \frac{r_1}{r_1+r_2}\}$.

However, when $r_1 < \gamma_1$, Player 1 can no longer guarantee $\max\{a_1 \in A^K \mid a_1 \leq \frac{r_2}{r_1+r_2}\}$. Rearranging the inequality, we have that Player 1 wins if

$$r_2(1-a_1) > (r_1 - \gamma_1)(1-a_2) + \gamma_1 a_1 \iff r_2 - (r_2+\gamma_1)a_1 > (r_1 - \gamma_1)(1-a_2).$$

Given that the right-hand side of the inequality is negative, but can be close to 0, Player 1 can guarantee winning by choosing any demand $a_1 \leq \frac{r_2}{\gamma_1+r_2}$.

Conversely, Player 2 can guarantee payoff $\frac{\gamma_1}{\gamma_1+r_2} - 1/K$ by choosing the demand $1-1/K$ (the inequality is flipped whenever a_1 is at least $\frac{r_2}{\gamma_1+r_2} + 1/K$). Observe that Player 2 guarantees this high payoff by choosing the greediest demand, which increases the disagreement D between the two players, which lowers concession rates λ_i and amplifies the disadvantage to Player 1. This is in contrast to prior results in the literature, in which players tend to make compromise demands to get their Rubinstein-like payoffs.

Note that the arguments above do not depend on the order of moves, so the limit payoffs in a rich type space are independent of the order of players' moves.

5. EXTENSIONS AND CONCLUSION

In this section, we describe additional extensions to showcase the applicability of our solution method and the robustness of our findings.

5.1. *Application: commitment to defend*

Countries form defense alliances (*e.g.* NATO and the Warsaw Pact) to publicly pledge to defend each other when they face an aggression. Similarly, to fight patent trolls who file frivolous infringement cases, big companies often follow through on the cases. And in recent years, teams

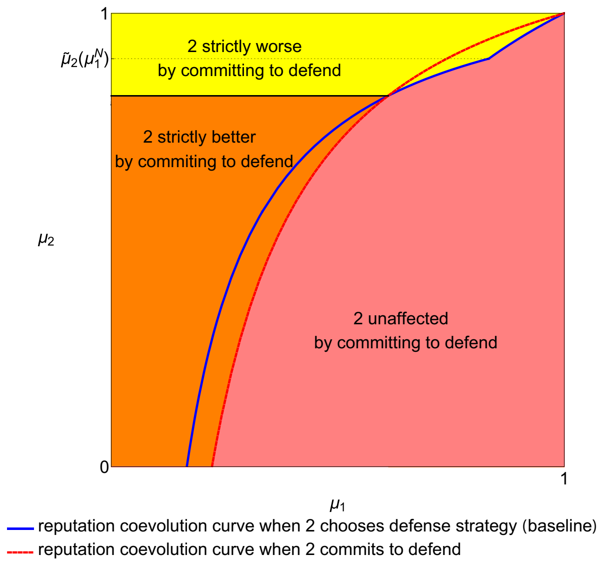

FIGURE 4

Comparison of reputation coevolution curves and equilibrium payoffs: 2 chooses defense strategy (baseline) versus 2 commits to defend

in MLB and the NHL pledge to go to the arbitration court when players ask for that, even though the two sides can continue to negotiate up to the arbitration date (usually 2–4 weeks from filing for arbitration). The formation of a defense alliance or a pledge to follow through in court or arbitration can be modelled in our setting as Player 2's commitment to see an ultimatum when Player 1 challenges. We can use the machinery we have developed to study the benefits and costs of this commitment. To facilitate exposition, we use terms in the context of defense alliance formation.

Consider the scenario in which Player 2 is ex ante committed to see any ultimatum. In this case, an unjustified Player 1 never challenges, because his payoff from challenge, $1 - a_2 - c_1 D$, is strictly worse than the payoff from concession, $1 - a_2$. This implies that the equilibrium play has a single strategy phase: Both players are indifferent to conceding at any time $t > 0$ and challenges are made by only a justified Player 1. The indifference conditions help pin down the equilibrium behaviour for both players.[15] The reputation coevolution curve is denoted by the dotted red curve in Figure 4.

Commitment to defend affects reputation building in two ways. First, commitment to defend has a deterrence effect on an unjustified Player 1. An unjustified Player 1 never challenges,

15. An unjustified Player 1 is indifferent between conceding at time t and conceding at time $t + dt$:

$$r_1(1 - a_2) = \widetilde{\lambda}_2(t) \cdot dt \cdot D \Rightarrow \widetilde{\lambda}_2(t) = \lambda_2.$$

An unjustified Player 2 is indifferent between conceding at time t and conceding at time $t + dt$:

$$r_2(1 - a_1) = \widetilde{\lambda}_1(t) \cdot dt \cdot D - \mu_1(t) \cdot \gamma_1 \cdot dt \cdot k_2 \cdot D \Rightarrow \widetilde{\lambda}_1(t) = \lambda_1 + \mu_1(t) \cdot \gamma_1 \cdot k_2.$$

Because of commitment to see a challenge, an unjustified Player 2 gets a strictly lower payoff than yielding to a challenge. However, Player 1's concession rate increases compared with the uncommitted benchmark to compensate for Player 2's payoff loss from commitment. Hence, players' reputations evolve according to $\dot{\mu}_1(t) = (\lambda_1 - \gamma_1) + \mu_1(t) \cdot \gamma_1 \cdot (1 + k_2)$ and $\dot{\mu}_2(t) = \lambda_2$.

so Player 1's reputation builds more slowly. Second, against a justified Player 1, commitment to defend leads to a loss for an unjustified Player 2. In equilibrium, Player 1 concedes faster to make up for Player 2's loss (to keep Player 2 indifferent across concession times), and this accelerates Player 1's reputation building.

In the baseline model, when Player 2's reputation is above $\widetilde{\mu}_2(\mu_1^N)$ (*i.e.* in the no-challenge phase), the deterrence effect against an unjustified Player 1 is also present, so commitment to defend brings no benefit but only loss against a justified Player 1. Hence, when players' reputations are close to 1, the reputation coevolution curve in the commitment case is above that in the no-commitment case. When $\mu_2 < \widetilde{\mu}_2(\mu_1^N)$, *i.e.* in the challenge phase in the baseline model, the deterrence effect is absent. In the commitment case the deterrence effect leads to Player 1's slower reputation building overall.[16] Hence, the reputation coevolution curve is flatter in the no-commitment case for $\mu_2 < \widetilde{\mu}_2(\mu_1^N)$. This explains why the two curves cross at most once.

Figure 4 illustrates the comparison between the reputation coevolution curve when 2 chooses defense strategy versus 2 commits to defend. Player 2 weakly benefits from a commitment to defend for any of her initial reputation that is not too large, and strictly benefits when in addition Player 1's initial reputation is sufficiently small. For the majority of instances when Player 2 strictly benefits, Player 1 is indifferent (the orange region to the left of the blue coevolution curve).

Player 2 is strictly worse off when her initial reputation is high and Player 1's reputation is low (the orange region). Note that Player 2 is worse off from committing to defend when an unjustified Player 1 would not have challenged in equilibrium without committing, *i.e.* when $\mu_2(t) > \widetilde{\mu}_2(\mu_1^N)$, because her commitment brings a lower payoff when she encounters a justified Player 1 but no benefit when she encounters an unjustified Player 1. In addition, however, Player 2 is also worse off for $\mu_2(t)$ close to and slightly below $\widetilde{\mu}_2(\mu_1^N)$.

The commitment to defend can be thought as the decision for Player 2 to join an alliance (*e.g.* NATO) in which countries commit to protect each other in case of aggression. Our model implies that there is benefit for joining such an alliance when one's own reputation is low, and there will be a strict benefit when the probability of a strong and dedicated rival is relatively low. It may not be beneficial to join a defensive alliance when a country's initial reputation to respond to aggression is high.

Can joining an alliance increase or decrease the total probability of conflict? On one hand, joining an alliance has a deterrence effect on unjustified opponents: The overall chance of challenge will decrease in equilibrium. On the other hand, joining an alliance increases the chance of conflict with justified opponents. The overall effect on the probability of conflict is ambiguous. This finding is consistent with and sheds light on the ongoing debate in the literature on the complicated relationship between defense alliances and conflict (Kenwick *et al.* 2015; Leeds and Johnson 2017; Morrow 2017).

Finally, note that in the limit case of rationality (*i.e.* when the initial probabilities of being justified are small), Player 2's commitment to defend does not affect players' payoffs.

5.2. *Extensions*

We summarise here our extensions detailed in Supplementary Material, Appendix C. First, to demonstrate the robustness of our findings to alternative specifications in the arrival of ultimatum

16. For $\mu_2 < \widetilde{\mu}_2(\mu_1^N)$, $\dot{\mu}_1^{\text{2commits}}(t) = \lambda_1 - \gamma_1 + \mu_1(t) \cdot \gamma_1 \cdot (1+k_2) < \dot{\mu}_1^{\text{baseline}}(t) = \lambda_1 - \gamma_1 + \mu_1(t) \cdot \frac{\gamma_1}{1-k_2}$.

opportunities, we consider the setting in which ultimatum opportunities arrive equally frictionally for both justified and unjustified players (Supplementary Material, Appendix C.1). In this setting, there may be an additional strategy phase in which an unjustified player's equilibrium ultimatum usage rate is capped by the frictional ultimatum opportunity arrival rate, which complicates equilibrium characterisation and uniqueness proof. Nonetheless, the frictional arrival of ultimatum opportunities does not alter our key qualitative results, such as discontinuous ultimatum and resolution rates, and payoffs in the limit case of rationality. In this setting, because an unjustified Player 1 cannot challenge more frequently than a justified Player 1, he never benefits from the introduction of ultimatum opportunities, which further highlights the importance of sufficiently frequent use of ultimatum opportunities to render not using them a sign of strength. This frictional ultimatum arrival setting enables us to compare private and public arrival of ultimatum opportunities. We demonstrate that the public arrival of ultimatum opportunities does not alter players' equilibrium payoffs: Compared with the baseline model, Player 1 challenges at a higher rate and concedes at a lower rate.

Moreover, we relax our assumption that justified players are committed by allowing strategic justified players (Supplementary Material, Appendix C.2). We then consider settings in which external resolution is costless, random, compromising, or noisy (Supplementary Material, Appendix C.3). These settings clarify the roles of bluffing opportunities and the external resolution mechanism in determining the bargaining behaviour and outcome, and further showcase the generality of our solution method to analyse alternative settings of conflicts.

Finally, we consider the extension in which both players have opportunities to challenge (Supplementary Material, Appendix C.4). If at least one player's exogenous ultimatum opportunity arrival rate is lower than the AG equilibrium concession rate, there exists a unique equilibrium outcome that is similar to the one in the setting with one-sided ultimatum opportunities. Otherwise, inefficient delays arise in equilibrium even in the limit case of rationality due to overabundant availability of access to external resolution opportunities. One implication of this result is that more convenient access to external resolution opportunities may be counterproductive and socially inefficient for resolution.

5.3. *Concluding remarks*

We study bargaining situations (1) that can be resolved not only internally but also externally (2) in which the outcome depends on parties' privately held information. Examples include patent infringement, labour disputes with arbitration, and negotiation with imminent war. Even when the external resolution does not favour unjustified players, they may nonetheless benefit from its availability: Although the potential arrival of ultimatum opportunities slows their reputation building, bluffing with an ultimatum when they have built a sufficiently high reputation may drive the opposing party to yield. In the limit in which the private information vanishes, immediate agreement and efficiency ensue, and determination of the winner and payoff division incorporates the ultimatum opportunity arrival rate in a parsimonious and intuitive manner. In addition, our model sheds light on the benefits and costs of defense alliance formation and the probability of war.

More questions are worth exploring. For example, we can model continuous-discrete-time games (Abreu and Pearce 2007) and study other equilibria, in which players' continuation payoffs after revealing rationality do not coincide with their concession payoffs. Another direction would be to include deadlines. Finally, we can study settings with nonstationary arrival of ultimatum opportunities or more complex demands such as nonstationary justified demands.

6. OMITTED PROOFS

Proof of Theorem 1. Let $\widehat{\Sigma} = (\widehat{\Sigma}_1, \widehat{\Sigma}_2) = ((\widehat{F}_1(\cdot), \widehat{G}_1(\cdot)), (\widehat{F}_2(\cdot), \widehat{q}_2(\cdot)))$ denote an equilibrium strategy profile. We argue that $\widehat{\Sigma}$ indeed define an equilibrium (which implies the existence of equilibrium strategies) and must have the form specified in the theorem (which implies the uniqueness of the equilibrium outcome). Let $u_i(t)$ denote the expected utility of an unjustified Player i who concedes at time t. Define $\mathcal{T}_i := \{t \mid u_i(t) = \max_s u_i(s)\}$ as the set of conceding times that attain the highest expected utility for Player i given opponent j's strategy $\widehat{\Sigma}_j$. Because $\widehat{\Sigma}$ is an equilibrium, $\mathcal{T}_i$ is nonempty for $i = 1, 2$. Furthermore, define $\tau_i := \inf\{t \geq 0 \mid \widehat{F}_i(t) = \lim_{s\to\infty} \widehat{F}_i(s)\}$ as the time of last concession for Player i, with $\inf \emptyset := \infty$. Finally, the support of Player 1's challenge distribution is $[0, \infty)$ due to the justified type's challenge behaviour. Hence, in any equilibrium, $\widehat{q}_2(t)$ maximises Player 2's expected payoff at time t when she faces a challenge when Player 1's reputation is $\nu_1(t)$ upon challenging, for almost every $t \leq \tau_2$ in both the $\widehat{G}_1$ measure and the Lebesgue measure. In the remainder of the proof, we will drop the "almost everywhere" qualifier. We obtain the following results.

(a) **Player 1's challenging strategy $\widehat{G}_1$ is continuous for $t \geq 0$.** To show that $\widehat{G}_1$ does not have any atoms, suppose to the contrary that $\widehat{G}_1$ jumps at time t so that an unjustified Player 1 challenges with a positive probability at time t; that is, $\widehat{G}_1(t) > 0$ for $t = 0$, or $\widehat{G}_1(t) - \widehat{G}_1(t^-) > 0$ for $t > 0$. Given that an unjustified Player 1 challenges with a positive probability and a justified Player 1 challenges with probability 0, when Player 2 faces a challenge she believes that a challenging Player 1 is unjustified with probability 1: $\nu_1(t) = 0$. Consequently, she is strictly better off responding to the challenge and obtaining a payoff of $1 - a_1 + (1 - w_1)D - k_2 D$ than yielding to the challenge and obtaining a payoff of $1 - a_1$, because $k_2 < 1 - w_1$ by assumption. But if Player 2 responds to a challenge with probability 1, an unjustified Player 1's payoff from challenging is less than $1 - a_1 + w_1 D - c_1 D$ (an unjustified Player 1's expected payoff when Player 2 who responds to a challenge is unjustified with probability 1), which is strictly less than his payoff from conceding. This is because $c_1 > w_1$ by assumption, so an unjustified Player 1 has a profitable deviation to conceding at t from challenging with a positive probability at t—a contradiction.

(b) **Player 2's yielding probability $\widehat{q}_2(t)$ is positive for almost all $t \leq \tau_2$.** Suppose to the contrary that $\widehat{q}_2(t) = 0$ on a set A of positive Lebesgue measure. Then $\int_A d\widehat{G}_1(t)dt = 0$. Then $\nu_1(t) = 1$ for almost every $t \in A$. Then $\widehat{q}_2(t) = 1$ for $t \in A$ is a profitable deviation—a contradiction.

(c) **Player 2's payoff when challenged at time t is $1 - a_1$ for almost all $t \leq \tau_2$.** Whenever an unjustified Player 2 yields to a challenge with a positive probability at time t in equilibrium, her payoff when being challenged at time t is equal to $1 - a_1$. By (*b*), Player 2 yields to a challenge with a positive probability for almost all $t \leq \tau_2$, so her payoff when challenged at time t is $1 - a_1$.

(d) **The last instant at which two unjustified players concede is the same: $\tau_1 = \tau_2$.** An unjustified player will not delay conceding upon learning that the opponent will never concede. Note that even if an unjustified Player 1 might challenge with a positive probability but never concedes, an unjustified Player 2's payoff from being challenged is $1 - a_1$ (by (*c*)), so she does not benefit from waiting for a challenge. Denote the last concession time by τ.

(e) **If $\widehat{F}_i$ jumps at t, then $\widehat{F}_j$ does not jump at t for $j \neq i$.** If $\widehat{F}_i$ has a jump at t, then Player j receives strictly higher utility by conceding an instant after t than by conceding exactly at t; note that whether or not Player 1 challenges at t does not affect the result, by (*c*).

(f) **If $\widehat{F}_2$ is continuous at time t, then $u_1(s)$ is continuous at $s = t$. If $\widehat{F}_1$ and $\widehat{G}_1$ are continuous at time t, then $u_2(s)$ is continuous at $s = t$.** These claims follow immediately from the definition of $u_1(s)$ in equation (1) and the definition of $u_2(s)$ in equation (2), respectively.

(g) **There is no interval $(t', t'') \subseteq [0, \tau]$ such that both $\widehat{F}_1$ and $\widehat{F}_2$ are constant on the interval (t', t'').** Assume the contrary and without loss of generality, let $t^* \leq \tau$ be the supremum of t'' for which (t', t'') satisfies the above properties. Fix $t \in (t', t^*)$ and note that for ε small enough there exists $\delta > 0$ such that $u_i(t) - \delta > u_i(s)$ for all $s \in (t^* - \varepsilon, t^*)$. In words, conditional on that the opponent is not conceding in an interval, it is strictly better for a player to concede earlier within that interval, and it is sufficiently significantly better by conceding early than by conceding close to the end of the time interval. By (e) and (f), there exists i such that $u_i(s)$ is continuous at $s = t^*$, so for some $\eta > 0$, $u_i(s) < u_i(t)$ for all $s \in (t^*, t^* + \eta)$ (observe that this relies on that Player 2 does not benefit from waiting for a challenge from Player 1, by (c)). In words, because of the continuity of the expected utility function at time t^*, the expected utility of conceding a bit after time t^* is still lower than the expected utility of conceding at time t within the time interval. Since $\widehat{F}_i$ is optimal, $\widehat{F}_i$ must be constant on the interval $(t', t^* + \eta)$. The optimality of $\widehat{F}_i$ implies that $\widehat{F}_j$ is also constant on the interval $(t', t^* + \eta)$, because Player j is strictly better off conceding before or after the interval than conceding during it. Hence, both functions are constant on $(t', t^* + \eta) \subseteq (t', \tau)$. However, this contradicts the definition of t^*.

(h) **If $t' < t'' < \tau$, then $\widehat{F}_i(t'') > \widehat{F}_i(t')$ for $i = 1, 2$.** If $\widehat{F}_i$ is constant on some interval, then the optimality of $\widehat{F}_j$ implies that $\widehat{F}_j$ is constant on the same interval, for $j \neq i$ (again, by (c)). However, (g) shows that $\widehat{F}_1$ and $\widehat{F}_2$ cannot be constant simultaneously.

(i) **$\widehat{F}_i$ is continuous for $t > 0$.** Assume the contrary: Suppose $\widehat{F}_i$ has a jump at time t. Then $\widehat{F}_j$ is constant on interval $(t - \varepsilon, t)$ for $j \neq i$. This contradicts (h).

(1) Strictly increasing $\widehat{F}_1$ and $\widehat{F}_2$ for $t < T$ follow from (h) and constant $\widehat{F}_1$ and $\widehat{F}_2$ for $t \geq T$ follow from (d).

(2) No atom for $\widehat{F}_i$ follows from (i). At most one atom for $\widehat{F}_1$ and $\widehat{F}_2$ at $t = 0$ follows from (e).

(3) (a) $\widehat{G}_1$ has no atom follows from (a), and (b) implies that $\widehat{G}_1$ is strictly increasing; if $\widehat{G}_1$ is constant, then $\widehat{q}_2(t) = 1$, which contradicts (b). (b) $\widehat{q}_2(t) \in (0, 1)$ for $t \in [0, T_1]$ follows from (b). From (f) and (i), it follows that $v_1(t)$ is continuous on $(0, \tau]$. Furthermore, $v_1(t)$ is strictly smaller than $1 - a_1$ when $\mu_2(t) > \mu_2^*$ (*i.e.* $\widehat{F}_2(t) > 1 - \frac{k_2}{1-z_2}$). Therefore, after $\mu_2(t) > \mu_2^*$, an unjustified Player 1 does not challenge. Since Player 2's reputation strictly increases over time, there is a finite time T_1 such that Player 1 challenges from time 0 to T_1 and does not challenge from T_1 onward. Hence, $\widehat{q}_2(t) = 0$ for $t \geq T_1$.

(4) It follows from (h) that $\mathcal{T}_i$ is dense in $[0, \tau]$ for $i = 1, 2$. From (d), (f), and (i), it follows that $u_i(s)$ is continuous on $(0, \tau]$, and hence $u_i(s)$ is constant for all $s \in (0, \tau]$. Consequently, $\mathcal{T}_i = (0, \tau]$. Hence, $u_i(t)$ is differentiable as a function of t and $du_i(t)/dt = 0$ for all $t \in (0, \tau)$.

In particular, Player 1's expected utility from conceding at time t is

$$u_1(t) = (1 - z_2)\int_0^t a_1 e^{-r_1 s} d\widehat{F}_2(s) + (1 - a_2)e^{-r_1 t}[1 - (1 - z_2)\widehat{F}_2(t)]. \tag{13}$$

The differentiability of $\widehat{F}_2$ follows from the differentiability of $u_1(t)$ on $(0, \tau)$. Differentiating equation (13) and applying Leibnitz's rule, we obtain

$$0 = a_1 e^{-r_1 t}(1 - z_2)\widehat{f}_2(t) - (1 - a_2)r_1 e^{-r_1 t}(1 - (1 - z_2)\widehat{F}_2(t)) - (1 - a_2)e^{-r_1 t}(1 - z_2)\widehat{f}_2(t),$$

where $\widehat{f}_2(t) = d\widehat{F}_2(t)/dt$. This in turn implies $\widehat{F}_2(t) = \frac{1-C_2 e^{-\lambda_2 t}}{1-z_2}$, where constant C_2 is yet to be determined. This characterisation implies that τ_2 is finite. At $\tau_1 = \tau_2$, optimality for Player i implies $\widehat{F}_1(\tau_1) + \widehat{G}_1(\tau_1) = 1$ and $\widehat{F}_2(\tau_2) = 1$.

This completes the proof that the structure of equilibrium strategies is unique. We now proceed to show the uniqueness of equilibrium strategies. We derive the reputation coevolution diagram using the reputation dynamics in Section 3.3. The reputation coevolution curve is strictly increasing, and $\widetilde{\mu}_1(\mu_2)$ is well defined for $\mu_2 \in (0, 1]$. Hence, the unique equilibrium entails $F_1(0) = 0$ and $\widehat{F}_2(0) > 0$ if $z_1 < \widetilde{\mu}_1(z_2)$; $\widehat{F}_1(0) > 0$ and $\widehat{F}_2(0) = 0$ if $z_1 > \widetilde{\mu}_1(z_2)$; and $\widehat{F}_1(0) = 0$ and $\widehat{F}_2(0) = 0$ if $z_1 = \widetilde{\mu}_1(z_2)$. Moreover, $F_1(0)$ is uniquely determined by equation (12), and $\widehat{F}_2(0)$ is uniquely determined analogously. This completes the uniqueness of equilibrium strategies. □

Proof of Proposition 1. We now consider a sequence of games in which all parameters of the game are fixed but the initial probabilities of commitment types, $\{z_1^n, z_2^n\}_n$, satisfy that $\lim \frac{z_1^n}{z_2^n} \in (0, \infty)$ and $\lim z_1^n = \lim z_2^n = 0$. Recall the reputation coevolution curve for $\mu_2 < \mu_2^F$,

$$\widetilde{\mu}_1(\mu_2 \mid \gamma_1) = \frac{\lambda_1 - \gamma_1}{\lambda_1(\mu_2)^{\frac{\gamma_1-\lambda_1}{\lambda_2}} + (\frac{\gamma_1}{\nu_1^*} - \gamma_1)(\frac{\mu_2}{\mu_2^*})^{\frac{\gamma_1-\lambda_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}}.$$

(i) If $\lambda_1 < \gamma_1$, then $\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1(\mu_2 \mid \gamma_1) = \nu_1^*(\gamma_1 - \lambda_1)/\gamma_1 = [1 - k_2/(1-w)](1 - \lambda_1/\gamma_1) > 0$. Therefore, in this case, along the equilibrium sequence of the sequence of games with vanishing probability of commitment types, Player 1 concedes at time 0 with a probability converging to 1 (since otherwise after time 0, the reputations would not land on the reputation coevolution diagram). Hence, we obtain efficiency in this case, where players agree on Player 2's terms right away—*i.e.* Player 2 is the "winner".

(ii) If $\lambda_1 = \gamma_1$, the expression of $\widetilde{\mu}_1(\mu_2 \mid \gamma_1 \neq \lambda_1)$ becomes

$$\widetilde{\mu}_1(\mu_2 \mid \gamma_1) = \begin{cases} \dfrac{1}{-\frac{\gamma_1}{\lambda_2}\log(\mu_2) + 1} & \text{if } \mu_2^* < \mu_2 < 1, \\ \dfrac{1}{-\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N} & \text{if } 0 < \mu_2 \le \mu_2^*, \end{cases}$$

where in this case $\mu_1^N = 1/[-\frac{\gamma_1}{\lambda_2}\log(\mu_2^*) + 1]$. Hence,

$$\begin{aligned}
\lim_{\mu_2 \to 0} \widetilde{\mu}_1'(\mu_2 \mid \gamma_1 \neq \lambda_1) &= \lim_{\mu_2 \to 0} \frac{\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2}}{\left[-\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N\right]^2} \\
&= \lim_{\mu_2 \to 0} \frac{-\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2^2}}{-2\left[-\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N\right]\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2}} \\
&= \lim_{\mu_2 \to 0} \frac{\frac{1}{\mu_2}}{2\left[-\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\log\left(\frac{\mu_2}{\mu_2^*}\right) + \mu_1^N\right]} \\
&= \lim_{\mu_2 \to 0} \frac{-\frac{1}{\mu_2^2}}{-2\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\frac{1}{\mu_2}} = \lim_{\mu_2 \to 0} \frac{1}{2\frac{\gamma_1}{\nu_1^*}\frac{1}{\lambda_2}\mu_2} = \infty,
\end{aligned}$$

where L'Hôpital's rule is applied once on each line. Hence, Player 2 will be the "winner".

(iii) If $\lambda_1 > \gamma_1$, then $\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1(\mu_2 \mid \gamma_1) = 0$. If $\lambda_1 > \gamma_1 + \lambda_2$, then $\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2 \mid \gamma_1) = 0$, if $\lambda_1 = \gamma_1 + \lambda_2$, then $\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2 \mid \gamma_1) > 0$, and if $\lambda_1 < \gamma_1 + \lambda_2$, then $\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2 \mid \gamma_1) = \infty$. The limits of $\widetilde{\mu}_1'(\mu_2 \mid \gamma_1)$ above can be derived from the expression of $\widetilde{\mu}_1(\mu_2 \mid \gamma_1)$ for $\mu_2 \le \mu_2^*$, which can be rearranged as

$$\widetilde{\mu}_1(\mu_2 \mid \gamma_1) = \frac{(\lambda_1 - \gamma_1)(\mu_2)^{\frac{\lambda_1 - \gamma_1}{\lambda_2}}}{\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}(\mu_2)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}.$$

The derivative is

$$\widetilde{\mu}_1'(\mu_2 \mid \gamma_1) = (\mu_2)^{\frac{\lambda_1-\gamma_1-\lambda_2}{\lambda_2}} \frac{\left[\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right](\lambda_1 - \gamma_1)}{\left[\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}} - \frac{\gamma_1}{\nu_1^*}(\mu_2)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}\right]^2},$$

which in the limit is

$$\lim_{\mu_2 \to 0^+} \widetilde{\mu}_1'(\mu_2 \mid \gamma_1) = \lim_{\mu_2 \to 0^+} (\mu_2)^{\frac{\lambda_1-\gamma_1-\lambda_2}{\lambda_2}} \frac{\lambda_1 - \gamma_1}{\lambda_1 + \gamma_1 \frac{1-\nu_1^*}{\nu_1^*}(\mu_2^*)^{\frac{\lambda_1-\gamma_1}{\lambda_2}}}.$$

The "winner" is Player 1 (resp., Player 2) if $\lambda_1 >$ (resp., $<$) $\gamma_1 + \lambda_2$, so there is efficiency. □

Proof of Proposition 2. Our result does not depend on the initial order of moves of the players in their demand choice. We will perform the analysis for the case in which Player 1 first picks a demand and then Player 2, observing this, chooses her demand, and then the war-of-attrition starts. Let $\sigma_1^n(i)$ be the equilibrium probability that Player 1 chooses type i/K in the nth game, and let $\sigma_2^n(j \mid i)$ be the equilibrium probability that Player 2 chooses type j/K after observing that Player 1 chooses i/K in the nth game. Let $(\sigma_1, \{\sigma_2(\cdot \mid i)\}_{i \in \{2,\ldots,K-1\}})$ be the limits of these strategies (along a convergent subsequence).

The first case is $\gamma_1 \le r_1$. In this case, if Player 1 chooses $a_1 = \max\{a \in A^K \mid a \le \frac{r_2}{r_1+r_2}\}$, then for any incompatible demand of Player 2, $\lambda_1 = \frac{r_2(1-a_1)}{a_1+a_2-1}$ is decreasing in a_2, so it is minimised at $a_2 = (K-1)/K$. In that case, $\lambda_1 > \gamma_1$. Hence, when Player 2 makes an incompatible demand, either $\sigma_2(\cdot \mid a_1) = 0$ or $\sigma_1(a_1) = 0$, and Player 1 is the winner, or the winner is determined by the comparison between $\lambda_1 - \gamma_1$ and λ_2.

$$\begin{aligned} \lambda_1 - \gamma_1 > \lambda_2 &\iff r_2(1-a_1) - \gamma_1(a_1 + a_2 - 1) > r_1(1-a_2) \\ &\iff r_2(1-a_1) - \gamma_1 a_1 > (1-a_2)(r_1 - \gamma_1). \end{aligned} \tag{14}$$

It is then routine to verify that if $a_1 = \max\{a \in A^K \mid a \le \frac{r_2}{r_1+r_2}\}$, and if $a_2 > 1 - a_1$, Player 1 is the winner.

Turning to Player 2 in this case, for any $a_1 > \frac{r_2}{r_1+r_2}$ such that $\sigma_1(a_1) > 0$, Player 2 is the winner if she demands $\max\{a \in A^K \mid a \le \frac{r_1}{r_1+r_2}\}$. This is again routine to verify. This completes the proof for $r_1 \ge \gamma_1$.

The second case is $\gamma_1 > r_1$. In this case, if Player 1 chooses $\max\{a \in A^K \mid a \le \frac{r_2}{\gamma_1+r_2}\}$, then for any incompatible demand of Player 2, $\lambda_1 > \gamma_1$. This is because λ_1 is decreasing in Player 2's demand, a_2, and when $a_2 < 1$ and when Player 1's demand is not more than $\frac{r_2}{\gamma_1+r_2}$, $\lambda_1 > \gamma_1$. Moreover, the right-hand side of equation (14), $(1-a_2)(r_1-\gamma_1) < 0$, and the left-hand side, $r_2(1-a_1) - \gamma_1 a_1 \ge 0$. Hence, whenever Player 2 chooses an incompatible demand a_2 with $\sigma_2(a_2 \mid a_1) > 0$, Player 1 is the winner. Hence, Player 1 secures the payoff of $\frac{r_2}{\gamma_1+r_2} - 1/K$.

Turning to Player 2 in this case, consider the strategy for Player 2 of always choosing $a_2 = (K-1)/K$. When Player 1's demand, a_1, is less than $\frac{r_2}{r_2+\gamma_1} + 1/K$, Player 2's payoff is at least $1 - a_1$, and our claim is true. If $a_1 \geq \frac{r_2}{r_2+\gamma_1} + 1/K$, and if $\sigma_1(a_1) > 0$, then

$$\lambda_1 = \frac{(1-a_1)r_2}{a_1 + a_2 - 1} = \frac{(1-a_1)r_2}{a_1 - 1/K} < \gamma_1,$$

which implies that Player 2 is the winner. Hence, Player 2 secures the payoff of $\frac{\gamma_1}{\gamma_1+r_2} - 1/K$. □

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Supplementary Data

Supplementary data are available at *Review of Economic Studies* online.

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