A Justification for Equal Sharing

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Abstract

This paper provides a justification for the persistence and prevalence of the equal sharing norm. In our framework, culturally differentiated agents endogenously form partnerships under different societal sharing norms, which vary in how output is divided. Although equal sharing is often viewed as less efficient than market based alternatives, we show that it can promote long-run efficiency by fostering the spread of cultural traits that support productive collaboration. In contrast, unequal sharing norms discourage such traits and hinder growth over time. Our findings provide a rationale for the prevalence of equal sharing in societies.

Keywords: Equal sharing, cultural norms, matching

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1 Introduction

The equal sharing norm advocates for the equitable distribution of resources among individuals. It has deep roots in human history and continues to influence contemporary societal structures. For example, the ancient principle of equal sharing in partnership law, as documented in the *Institutes of Justinian* since 533 CE, remains widely adopted by modern law firms (Smith, 2001). Equal sharing of loyalties among band members is a common practice, even among the most renowned bands (Polcz, 2023). Kibbutz has been a vibrant Israeli community that promotes equality of resource allocation among community members (Abramitzky, 2008, 2011, 2018).

While equal sharing offers appealing benefits, such as fostering fairness and reinforcing group cohesion, it is inherently inflexible. Market-based approaches, which allow for compensation to vary based on different factors, are often seen as more efficient and adaptable to changing conditions. Despite this, equal sharing remains a common practice in certain areas of the market economy. This raises the question—what underlying mechanisms sustain the persistence of equal sharing, even when more flexible alternatives are available? We explore this question in this paper.

We model a heterogeneous population of agents who differ in cultural traits and endogenously form partnerships. The output of each partnership depends on the traits of the matched agents. To fix ideas, consider three examples. First, agents may differ in their level of generalized trust (the belief that most people in society can be trusted). Partnerships involving highly trusting agents are more productive, as trust reduces ex ante contracting and monitoring costs and allows partners to focus on joint output rather than guarding against opportunism. Second, agents may differ in patience. When both partners are future-oriented, they are more willing to incur upfront costs, which is important for the long-term success of their collaboration through repeated interactions. Third, agents may differ in risk attitudes. In this case, the most productive partnerships may instead involve a mix of traits: one partner is willing to pursue high-variance opportunities, while the other plays a more cautious role, identifying potential pitfalls and constructing safeguards. Such pairings can achieve better outcomes than homogeneous matches. The first two examples illustrate when partnership output is increasing in the individual traits of both partners, and the last example shows when differences in traits across partners can lead to higher joint productivity.

The resulting output within each match is then divided according to prevailing societal sharing norms. We analyze three types of sharing norms. First, the ***market sharing norm*** allows agents to split output based on their market opportunities, as in a perfectly transferable utility (TU) matching market. Second, the ***equal sharing norm*** requires matched pairs to divide output equally. Third, we consider ***unequal sharing norms***, such as those that allocate a larger (or smaller) share to the more senior agent. Unlike the market sharing norm, equal and unequal sharing norms lack flexibility, causing the matching market to operate as a nontransferable utility (NTU) one.

In our model, there are two tiers of evolution: matching behavior and culture. At the behavioral level, agents within a population engage in a sequence of matching stages in which each stage immediately results in a stable matching. At the cultural trait level, the distribution of traits evolves more slowly across stages over time, driven by the payoffs earned through matching within each stage, and converges to a stable distribution in the long run.

We find that while the equal sharing norm may not yield as efficient stable matchings as the market-sharing norm at the behavioral level, both lead to efficiency at the level of cultural evolution. The mechanisms underlying the equal sharing norm and the market based sharing norm differ in important ways. The equal sharing norm reflects an egalitarian commitment to equal outcomes, while the market based norm emphasizes equal opportunities and rewards based on contribution. Equal sharing tends to encourage the formation of productive matches, even when traits are complementary or unequally distributed. In contrast, market based sharing does not necessarily maximize the number of productive partnerships, but it ensures that advantageous traits are rewarded. In this sense, both mechanisms support the emergence of cultural traits that enhance efficient collaboration, albeit through distinct channels.

In contrast, unequal sharing norms can introduce inefficiencies in the process of cultural evolution, as traits conducive to productivity may be suppressed when they are disadvantaged within a hierarchical society. This suggests that, over time, unequal sharing norms can be outperformed by both the market sharing norm and the equal sharing norm, which may coexist in the long run.

Our results shed light on the broader discussion of how differences in social structures contribute to the persistent variation in cultural traits across societies. For example, they speak to why historically the West and the East tend to produce and sustain different trust patterns. Greif and Tabellini (2010, 2017) argue that Western Europe developed relatively egalitarian and impersonal forms of social organizations—including corporations, guilds, self-governing towns, and looser kinship structures—while China maintained hierarchical, lineage-based systems centered on extended family networks. Alesina and Giuliano (2015), in a comprehensive review, document that weaker family ties and lower power distance—both features of Western societies—are strongly associated with higher levels of generalized trust. In our model, matching occurs primarily among strangers at large in society, making generalized trust essential for successful partnerships. This suggests that the more egalitarian social norms found in Western societies—whether supported by the equal sharing norm or the market based norm—help create conditions that favor the growth of trust and generalized, whereas more hierarchical societies, such as those in East Asia, are less conducive to its development.

Previous studies adopt a similar approach to study the evolution of cultural or preference traits under rapid matching and a fixed sharing rule (e.g., Wu and Zhang, 2021; Hiller, Wu, and Zhang, 2023; Wang and Wu, 2025). However, they do not investigate how different sharing rules shape long-run outcomes differently.

In our model, partnership formation happens in a one-sided pairwise matching market (i.e., roommate market) with a continuum of agents in every period. Although many standard results from the parallel two-sided matching settings—e.g., equivalence of socially efficient and stable matchings, convexity of stable payoffs (Shapley and Shubik, 1972; Gretsky, Ostroy, and Zame, 1992)—extend to our one-sided setting, there is no paper that explicitly states and proves these results.[1] To this end, we provide results in more detail and prove them rigorously, which may serve as a reference for subsequent papers that model the evolution of behavior, productivity, culture,

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[1] A few papers—Azevedo, Weyl, and White (2013); Chiappori, Salanié, and Galichon (2019); Carmona and Laohakunakorn (2024)—provide existence results in similar roommate matching settings, but do not discuss properties such as convexity and compactness of the set of stable payoffs that are crucial to our evolutionary existence results.

and/or preferences in one-sided matching markets. In addition, it should be noted that although we model explicitly the formation of two-member teams, the main result of the long-run efficiency of the equal sharing norm would go through under the formation of multiple-member teams.

The only other work we are aware of that provides evolutionary justifications for the equal sharing norm is the book of Skyrms (2014). In his model, a population of agents is randomly matched to play a dollar division game. The population consists of three types of agents: super-greedy agents who demand at least 2/3, fair-minded agents who demand at least 1/2, and super-modest agents who demand at least 1/3. The author demonstrates that, under replicator dynamics, a homogeneous population of fair-minded agents as well as a polymorphic population with equal proportions of super-greedy and super-modest agents are both asymptotically stable. However, the basin of attraction for the fair-minded population is significantly larger than that of the polymorphic population. The intuition is straightforward. Super-greedy agents secure 2/3 only when paired with super-modest agents and receive nothing otherwise. Super-modest agents can always reach an agreement with any opponent, but they consistently receive the smallest share. Fair-minded agents, by contrast, typically secure 1/2, especially when there is a sufficient number of fair-minded or super-modest agents in the population.

Our model differs from Skyrms (2014) in several key aspects. First, agents in our model are heterogeneous in their cultural traits that affect the productivity of partnerships, rather than in demand rules. Second, partnerships in our model emerge endogenously, as opposed to being formed through random encounters, and this is key to the long-run efficiency of the equal sharing norm in our evolutionary setting.

The rest of the paper is organized as follows. Section 2 studies the market sharing norm. Section 3 studies alternative fixed-proportion sharing norms, i.e., equal and unequal sharing norms. Section 4 provides an additional discussion of how sharing norms evolve. Section 5 concludes. The appendix collects omitted proofs.

2 The Market Sharing Norm

2.1 Population and Matching

Consider a unit mass of agents. Each agent is characterized by a trait $i \in N = \{1, 2, ..., n\}$. Let $x_i \in [0, 1]$ denote the mass of trait-i agents, for any $i \in N$, and $\sum_{i=1}^{n} x_i = 1$. Let $x = (x_1, x_2, ..., x_n)$ denote the population state. In this paper, we use $\mathbb{R}_+$ to refer to the set of nonnegative real numbers and $\mathbb{R}_{++}$ to refer to the set of strictly positive real numbers. Let $X = \{x \in \mathbb{R}_+^n | \sum_{i=1}^{n} x_i = 1\}$ denote the set of population states; it is an $(n-1)$-simplex Δ^{n-1}. Let $X_{++} = \text{int}(X) = \{x \in \mathbb{R}_{++}^n | \sum_{i=1}^{n} x_i = 1\}$ denote the ***interior*** of X, i.e., the set of population states such that there is a positive mass of each trait.

Agents can find partners to form pairs; they can also choose not to: If an agent remains unmatched, she receives a payoff of 0. If a trait-i agent and a trait-j agent are matched, they jointly produce a strictly positive payoff of $f(i, j)$. Let $f : N^2 \to \mathbb{R}_{++}$ denote the production function. Assume that f is symmetric: $f(i, j) = f(j, i)$ for any $i, j \in N$.

Let $\mu_{ij}(x) \geq 0$ denote the mass of (i, j) pairs given the population state x. We have $\mu_{ij}(x) = \mu_{ji}(x)$. Since each (i, i) pair contains two trait-i agents, and each (i, j) pair with $i \neq j$ contains one trait-i agent, we have $2\mu_{ii}(x) + \sum_{j \neq i} \mu_{ij}(x) \leq x_i$. Let $\mu(x)$ be an $n \times n$ matrix that denotes the aggregate matching (Echenique et al., 2013), whose (i, j) entry is $\mu_{ij}(x)$. Note that the notion of aggregate matching is defined on the population level as opposed to the more familiar individual level. We adopt the stability concept in two-sided TU (transferable utility) matching market (Shapley and Shubik, 1972; Becker, 1973; Gretsky, Ostroy, and Zame, 1992) to our setting of one-sided matching market with a continuum of agents to characterize the stable outcome of the partnership formation process.

Definition 1 *A* ***stable outcome*** *is described by a* ***stable matching*** $\mu^*(x)$ *and a vector of* ***stable payoffs*** $w^*(x) = (w_1^*(x), w_2^*(x), ..., w_n^*(x)) \in \mathbb{R}^n$*, such that*

*(i) (**individual rationality**) each person gets at least as much as staying unmatched:* $w_i^*(x) \geq 0$ *for any* $i \in N$*;*

(ii) ***(surplus efficiency)*** *each pair exactly divides the surplus:* $w_i^*(x) + w_j^*(x) = f(i,j)$ *if* $\mu_{ij}^*(x) > 0$*, for any* $i, j \in N$*; and*

(iii) ***(no blocking pair)*** *no pair can get a sum that is strictly more than the sum of their current payoffs:* $w_i^*(x) + w_j^*(x) \geq f(i,j)$ *for any* $i, j \in N$*.*

Let us note some implications of the definition. First, since an agent gets a payoff of zero if she remains unmatched and would produce a strictly positive surplus with any other agent, all agents are matched in any stable matching. Second, in the definition, all agents with the same trait have the same stable payoff. This is not an assumption, but would be implied by the definition, namely, by the no-blocking-pair condition: If two agents with the same trait have different payoffs, then there is room for a blocking pair to emerge.

Lemma 1 *A stable outcome exists.*

We show in the proof of Lemma 1 that just as in a two-sided matching market (Gretsky, Ostroy, and Zame, 1992), a matching is stable if and only if it is socially efficient. Hence, we can characterize a stable matching by the solution to the maximization problem of the total payoff:

$$\max_{\mu(x)} \sum_{i\in N} \sum_{j \leq i} \mu_{ij}(x) f(i,j) \text{ such that } 2\mu_{ii}(x) + \sum_{j \neq i} \mu_{ij}(x) \leq x_i \text{ for all } i \in N.$$

When the solution to the maximization problem is unique, it must be the unique stable matching. We will assume that the stable matching is always unique in the subsequent analysis.[2] Let $W^*(x)$ be the collection of stable payoff vectors $w^*(x)$ that support the unique stable matching $\mu^*(x)$.

As an example, consider the setting with two traits, $N = \{1, 2\}$. First, suppose $2f(1,2) < f(1,1) + f(2,2)$. The unique stable outcome involves positive assortative matching—$\mu_{ii}^* = x_i/2$ for $i \in N$—and equal split—$w_i^*(x_i) = f(i,i)/2$ for $i \in N$. Next, suppose instead $2f(1,2) > f(1,1) + f(2,2)$. The stable outcome involves negative assortative matching. When $x_1 < 1/2$,

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[2] A sufficient condition to ensure unique stable outcome is $f(i,j)+f(i',j') \neq f(i,i')+f(j,j')$ for any $i, i', j, j' \in N$ such that $\{i,i'\} \cap \{j,j'\} = \emptyset$.

the unique stable matching is given by $\mu^*_{12}(x) = x_1$ and $\mu^*_{22}(x) = x_2 - x_1$, and the unique stable payoff vector is given by $w^*(x) = (f(1,2) - f(2,2)/2, f(2,2)/2)$. When $x_1 > 1/2$, the unique stable matching is given by $\mu^*_{12}(x) = x_2$ and $\mu^*_{11}(x) = x_1 - x_2$, and the unique stable payoff vector is given by $w^*(x) = (f(1,1)/2, f(1,2) - f(1,1)/2)$. When $x_1 = 1/2$, the unique stable matching $\mu^*(x)$ is given by $\mu^*_{12}(x) = 1/2$, and any payoff vector $w^*(x)$ such that $f(i,i)/2 \leq w^*_i(x) \leq f(1,2) - f(j,j)/2$ for $i \in \{1,2\}$ and $j \neq i$, and $w^*_1(x) + w^*_2(x) = f(1,2)$ is stable.

2.2 The Evolution of Traits

The distribution of traits evolves over time, guided by the payoffs associated with different traits. We assume that the process of forming partnerships reaches a stable outcome much more rapidly than the evolution of traits. Thus, when the distribution of traits shifts, the stable outcome swiftly adapts.[3]

We consider the replicator dynamic to describe the evolution of traits. The replicator dynamic can be derived from a biological growth model (Taylor and Jonker, 1978), an intergenerational cultural transmission model (Bisin and Verdier, 2001), or learning models based on imitation (Sandholm, 2010). Let $\overline{w}^*(x) = \sum_{j\in N} x_j w^*_j(x)$ denote the average payoff in a stable outcome. Note that since there is a unit mass of agents, the average payoff equals the total payoff: $\overline{w}^*(x) = \sum_{i\in N}\sum_{j\leq i} \mu^*_{ij}(x) f(i,j)$. Since the unique stable matching $\mu^*(x)$ must maximize the total payoff, $\overline{w}^*(x)$ does not vary across $w^*(x) \in W^*(x)$. Let $\overline{\mathbf{w}}^*(x) = (\overline{w}^*(x), \overline{w}^*(x), ..., \overline{w}^*(x)) \in \mathbb{R}^n$ be the n-dimensional vector in which each of the n entries equals $\overline{w}^*(x)$. The replicator dynamic is given by a differential inclusion:[4]

$$\dot{x} \in V^{RDI}(x) \equiv \left\{ x \cdot [w^*(x) - \overline{\mathbf{w}}^*(x)] \,|w^*(x) \in W^*(x) \right\}. \tag{RDI}$$

Hence, the fraction of agents who carry a trait that is associated with an above-average payoff

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[3] This approach parallels the indirect evolutionary approach that studies preference evolution (Güth and Yaari, 1992; Güth, 1995), where agents' preferences evolve at a much slower pace compared to their behaviors.

[4] A differential inclusion is a generalization of a differential equation where the derivative of a function is constrained to belong to a set of possible values rather than being determined by a single value.

grows, and vice versa.

The dynamic described in (RDI) is a differential inclusion because of the possible indeterminacy of stable payoff vectors. When there are multiple stable payoff vectors, the path the dynamic takes depends on the realized stable payoff vector. For example, at x, if the realized stable payoff vector is $w^*(x)$, then the growth rate of trait i is given by $\dot{x}_i = x_i \cdot [w_i^*(x) - \overline{w}^*(x)]$ for any $i \in N$, which takes the standard form of the replicator dynamic.

We have the following theorem from Sandholm (2010) that provides sufficient conditions for the existence of a solution trajectory.

Theorem 1 *(Sandholm, 2010, Theorem 6.A.1) Consider a differential inclusion $\dot{x} \in V(x)$. If V is nonempty, bounded, convex-valued, and upper-hemicontinuous, then for any initial condition $\xi \in \mathbb{R}^n$, there exists a* ***Carathéodory solution trajectory*** *$\{x^t\}_{t\in[0,T]}$ to the differential inclusion. That is, the solution trajectory is Lipschitz continuous, and its derivative $\dot{x}_t \in V(x)$ at almost all times $t \in [0, T]$.*

The following lemma shows that the replicator dynamic we consider is well-behaved and thus, combined with Theorem 1, has a solution trajectory in the interior of X. Note that the uniqueness of the solution trajectory is not guaranteed.

Lemma 2 *The correspondence V^{RDI} is nonempty-valued, bounded, convex-valued, and upper-hemicontinuous in X_{++}.*

2.3 Analysis

We now study convergence and stability of the dynamic processes. Convergence informs us where the dynamic process ends up from all initial conditions, and stability determines whether convergent states of the population are robust against perturbations.

Definition 2 *The* ***ξ-limit set*** *$\omega(\xi)$ is the set of all population states such that the solution trajectory*

$\{x^t\}_{t\geq 0}$ *starting from* $x^0 = \xi$ *approaches arbitrarily closely infinitely often. Formally,*

$$\omega(\xi) := \left\{ y \in X : \exists \{t_k\}_{k=1}^{\infty} \text{ with } \lim_{k\to\infty} t_k = \infty \text{ s.t. } x^0 = \xi \text{ and } \lim_{k\to\infty} x^{t_k} = y \right\}.$$

An element in a ξ*-limit set is called a* ***ξ-limit point****.*

The collection of all limit points is denoted by the set $\Omega = \cup_{\xi\in X}\omega(\xi)$: It captures the notion of recurrence of the dynamics. Let $A \subseteq X$ be a closed set, and call $O \subseteq X$ a neighborhood of A if it is open relative to X and contains A.

Definition 3 *Set* $A \subset X$*, a proper subset of* X*, is* ***Lyapunov stable*** *if for every neighborhood O of A, there exists a neighborhood O′ of A such that every solution trajectory* $\{x^t\}_{t\geq 0}$ *that starts in O′ is contained in O, that is,* $x^0 \in O'$ *implies that* $x^t \in O$ *for all* $t \geq 0$*. If A is not Lyapunov stable, we call it* ***Lyapunov unstable****.*

Intuitively, Lyapunov stability requires that any displacement from A does not lead the process to go “far” from A at any point in time.

Definition 4 *Set* $A \subset X$ *is* ***attracting*** *if there is a neighborhood* $Y \subseteq X$ *of A such that every solution trajectory that starts in* Y *converges to A, that is,* $\xi \in Y$ *implies* $\omega(\xi) \subseteq A$*. Set* $A \subset X$ *is* ***globally attracting*** *if it is attracting with* $Y = X$*. The set of points* $\xi \in X$ *such that* $\omega(\xi) \subseteq A$ *is called the* ***basin of attraction*** *of A.*

Intuitively, that A is attracting requires that given any displacement from A, the process returns to A in the limit.

Definition 5 *Set* $A \subset X$ *is* ***asymptotically stable*** *if it is Lyapunov stable and attracting. Set* $A \subset X$ *is* ***globally asymptotically stable*** *if it is Lyapunov stable and globally attracting.*

Intuitively, asymptotic stability of A requires that given any displacement from A, the process never travels “far” from A and returns to A in the limit. The limiting behavior of deterministic dynamics can be characterized as follows. We will focus on the case that there exists a unique match that leads to the highest payoff.

Proposition 1 ***(i)*** *Suppose there exists a unique* i *such that* $f(i,i) = \max_{k,l} f(k,l)$. *The population state* $x^* = e_i$, *the* i*th standard basis vector, is the unique globally asymptotically stable state under the replicator dynamic with respect to* $X \setminus \{x | x_i = 0\}$. *It is also the efficient state.* ***(ii)*** *Suppose there exist* i *and* $j \neq i$ *such that* $f(i,j) = \max_{k,l} f(k,l)$. *The population state* x^* *such that* $x_i^* = x_j^* = 0.5$ *is the unique globally asymptotically stable state under the replicator dynamic with respect to* $X \setminus \{x | x_i = 0$ *or* $x_j = 0$ *or both*$\}$. *It is also the efficient state.*

Proposition 1 shows that the population states that maximize the average payoff in stable matching are asymptotically stable. See Appendix A for Theorems 2 and 3 that we use to prove stability and convergence based on the Lyapunov method. In the first case when there exists a unique i such that $f(i,i) = \max_{k,l} f(k,l)$, we define $L^{RDI}(x) = -\log x_i$. In the second case when there exist i and $j \neq i$ such that $f(i,j) = \max_{k,l} f(k,l)$, we define $L^{RDI}(x) = -[\log(2x_i) + \log(2x_j)]$. We will use $L^{RDI}(x)$ as the Lyapunov function for (RDI).

To make Proposition 1 more concrete, consider the setting with two cultural traits. Without loss of generality, assume that $f(1,1) > f(2,2)$. Hence, the two traits differ in own-match productivity. When $2f(1,2) > f(1,1) + f(2,2)$, we call the payoff structure ***submodular***. There is always a unique stable matching, in which the mass of cross-trait matching is maximized (negative assortative matching). When $2f(1,2) < f(1,1) + f(2,2)$, we call the payoff structure ***supermodular***. There is always a unique stable matching with only own-trait matching (positive assortative matching).

Case 1. Suppose $f(1,2) > f(1,1)$. This immediately implies submodularity. In this case, we always have a unique stable payoff as long as $x_1 \neq 1/2$. When $x_1 < 1/2$, $w_1(x) = f(1,2) - f(2,2)/2 > w_2(x) = f(2,2)/2$. When $x_1 > 1/2$, $w_1(x) = f(1,1)/2 < w_2(x) = f(1,2) - f(1,1)/2$. Hence, for (RDI), $\dot{x}_i > 0$ when $0 < x_1 < 1/2$; $\dot{x}_i < 0$ when $x_1 > 1/2$, which guarantees the globally asymptotic stability of $x^* = (1/2, 1/2)$ given any interior initial state.

Case 2. Suppose $f(1,1) > f(1,2)$ and $2f(1,2) > f(1,1) + f(2,2)$. We still have submodularity. When $x_1 < 1/2$, $w_1(x) = f(1,2) - f(2,2)/2 > w_2(x) = f(2,2)/2$. When $x_1 > 1/2$, $w_1(x) = f(1,1)/2 > w_2(x) = f(1,2) - f(1,1)/2$. Hence, for (RDI), $\dot{x}_i > 0$ when $x_1 > 0$, which guarantees the globally asymptotic stability of $x_1 = 1$ in $(0, 1]$.

Case 3. Suppose $f(1,1) > f(1,2)$ and $2f(1,2) < f(1,1) + f(2,2)$. In this case, we have supermodularity. We always have $w_1(x) = f(1,1)/2 > w_2(x) = f(2,2)/2$. Hence, for (RDI), $\dot{x}_i > 0$ when $x_1 > 0$, which guarantees the globally asymptotic stability of $x_1 = 1$ in $(0, 1]$.

To summarize this two-trait example, while supermodularity guarantees that trait 1, the more productive cultural trait in terms of own-trait matching, prevails through evolution, submodularity does not guarantee the coexistence of both traits. Only when cross-trait match leads to the highest production can both traits coexist. These results confirm Proposition 1, the population state that maximizes the average payoff in stable matching is the long-run prediction of the evolutionary process. When $f(1,2)$ is the highest, $x = (1/2, 1/2)$ maximizes the average payoff. When $f(1,1)$ is the highest, $x = (1, 0)$ maximizes the average payoff.

3 Fixed-Proportion Sharing Norms

In this section, rather than letting agents determine surplus division endogenously within the market, we introduce exogenous sharing norms that specify how the surplus is distributed in each match. These norms reflect the social conventions that governed resource allocation in society before the emergence of free markets.

Let u_{ij} denote a trait-i agent's payoff from her match with a trait-j agent. $u_{ij} + u_{ji} = f(i, j)$. We adopt the stability concept in two-sided NTU (nontransferable utility) matching market (Gale and Shapley, 1962) to our setting of one-sided matching markets with a continuum of agents to characterize the stable matching of the partnership formation process.

Definition 6 *A **stable matching** $\mu^*(x)$ satisfies*

*(i) (**individual rationality**) each person gets at least as much as staying unmatched: $u_{ij} \geq 0$ for any $i, j \in N$ such that $\mu^*_{ij}(x) > 0$;*

*(ii) (**no blocking pair**) no pair of persons can both be strictly better off than what they have in*

their current matches: $u_{ij} \leq u_{ik}$ *or* $u_{ji} \leq u_{jl}$ *for any* $i, j, k, l \in N$ *such that* $\mu^*_{ik}(x) > 0$ *and* $\mu^*_{jl}(x) > 0$.

Carmona and Laohakunakorn (2024) prove the existence of stable matching. For x and $\mu^*(x)$, let $u^*_i(x)$ denote the average payoff of all trait-i agents. Let $u^*(x) = (u^*_1(x), u^*_2(x), ..., u^*_n(x)) \in \mathbb{R}^n$ denote the vector of average payoffs. Let $\overline{u}^*(x)$ denote the average payoff in the entire population. Let $\overline{\mathbf{u}}^*(x) = (\overline{u}^*(x), \overline{u}^*(x), ..., \overline{u}^*(x)) \in \mathbb{R}^n$. The evolution of traits is still dictated by the replicator dynamic. But now it is given by a differential equation:

$$\dot{x} = V^{RDE}(x) \equiv \{x \cdot [u^*(x) - \overline{\mathbf{u}}^*(x)]\}. \tag{RDE}$$

Theorem 4 in Appendix A establishes asymptotic stability for (RDE).

3.1 Equal Sharing Norm

First, suppose two agents in a pair equally split the payoff they jointly produce. That is, $u_{ij} = f(i,j)/2$. In the remaining analysis, we will assume that $f(i,j) \neq f(i,l)$, for any i and any $j \neq l$ (see footnote 2). Without loss of generality, assume $f(1,1) > f(i,i)$ for any $i \neq 1$.

Proposition 2 ***(i)*** *Suppose* $f(1,1) = \max_{k,l} f(k,l)$. *Then* $x^* = e_1$, *where* e_1 *is the first standard basis vector, is globally asymptotically stable under* (RDE) *with respect to* $X \backslash \{x | x_1 = 0\}$. *It is also the efficient state.* ***(ii)*** *Suppose there exist* i *and* j *such that* $i \neq j$ *and* $f(i,j) = \max_{k,l} f(k,l)$. *Then* x^* *such that* $x^*_i = x^*_j = 1/2$ *is globally asymptotically stable under* (RDE) *with respect to* $X \backslash \{x | x_i = 0$ *or* $x_j = 0$ *or both*$\}$. *It is also the efficient state.*

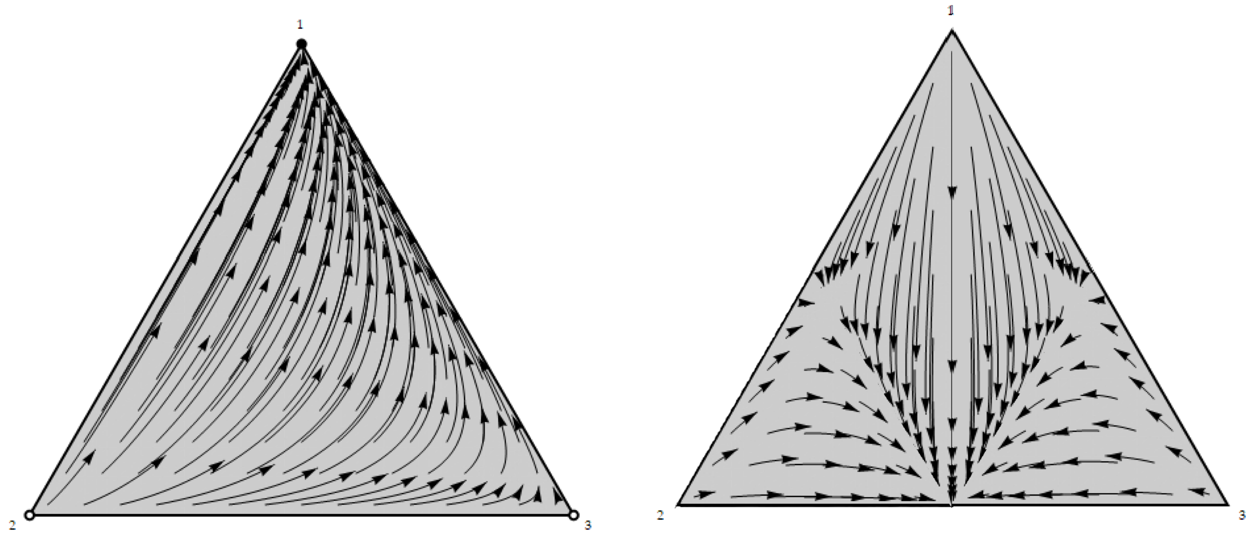
Similar to the analysis for the market sharing norm, here we will focus on two cases. In the first case when a single trait (trait 1) is the most productive in the sense that two trait-1 agents generate the highest surplus, we define $L^{RDE}(x) = -\log x_1$. In the second case where there exist i and j such that $i \neq j$ and $f(i,j) = \max_{k,l} f(k,l)$, we define $L^{RDE}(x) = -[\log(2x_i) + \log(2x_j)]$. Proposition 2 will use $L^{RDE}(x)$ as the Lyapunov function for (RDE).

Note that $L^{RDE}(x)$ is identical to $L^{RDI}(x)$, which we use to prove stability for (RDI) in Section 2.3. Although they are the same, they function differently in the two settings due to the inherent differences between TU and NTU matching. Specifically, when a pair of traits (including potentially the same trait) are most productive when matched, these traits will pair with each other to the greatest extent possible under NTU matching, as in the equal sharing norm setting. This results in their average payoff being higher than the population average—an essential condition for the Lyapunov method to apply. However, under TU matching, as in the market sharing norm setting, these traits may not necessarily pair with each other to the greatest extent possible. Nevertheless, the no-blocking-pair condition under TU matching still ensures that the average payoff for these traits exceeds the population average.

To illustrate, we consider settings with three traits. Suppose $f(1,1) > f(1,2)$, $f(1,1) > f(1,3)$, $f(2,2) > f(1,2)$, $f(2,2) < f(2,3)$, $f(3,3) < f(1,3)$, and $f(3,3) > f(2,3)$. This payoff structure leads to PAM and the globally attracting and stable state $x^* = (1,0,0)$. Figure 1a provides a graphic illustration of (RDE).

Alternatively, suppose $f(1,1) < f(1,2) < f(1,3)$, $f(2,2) < f(1,2) < f(2,3)$, and $f(3,3) < f(1,3) < f(2,3)$. Suppose the initial state $x^0 = (x_1^0, x_2^0, x_3^0) = (0.3, 0.5, 0.2)$. The unique stable matching is characterized by $\mu_{12} = 0.3, \mu_{23} = 0.2$. Then the payoffs under stable matching are $u_1^*(x^0) = f(1,2)/2$, $u_2^*(x^0) = 0.6 \cdot [f(1,2)/2] + 0.4 \cdot [f(2,3)/2]$ and $u_3^*(x^0) = f(2,3)/2$. The average payoff is $\overline{u}^*(x^0) = 0.6 \cdot [f(1,2)/2] + 0.4 \cdot [f(2,3)/2]$. Hence, under the replicator dynamic, $\dot{x}_1 = x_1^0 \cdot [u_1^*(x^0) - \overline{u}^*(x)] < 0$, $\dot{x}_2 = x_2^0 \cdot [u_2^*(x^0) - \overline{u}^*(x)] = 0$, $\dot{x}_3 = x_3^0 \cdot [u_3^*(x^0) - \overline{u}^*(x)] > 0$. Hence, x_1 shrinks, x_2 stagnates, and x_3 grows. Therefore, $\lim_{t\to\infty} x^t = (0, 0.5, 0.5)$. Figure 1b provides a graphic illustration of (RDE).

Figure 1: Comparison of replicator dynamics under PAM and Non-PAM.



(a) Replicator dynamic under PAM.

(b) Replicator dynamic under Non-PAM.

Note: Panel (a) illustrates the replicator dynamics when $f(1,1) = 6$, $f(1,2) = 1$, $f(1,3) = 5$, $f(2,2) = 2$, $f(2,3) = 3$, and $f(3,3) = 4$. Panel (b) illustrates the replicator dynamics when $f(1,1) = 1$, $f(1,2) = 3$, $f(1,3) = 5$, $f(2,2) = 2$, $f(2,3) = 6$, and $f(3,3) = 4$.

In sum, under the equal sharing norm, efficiency is always achieved through evolution for any payoff structure f.

3.2 Unequal Sharing Norms

Now suppose the society has a universal sharing norm that features an unequal division of joint production. Let $\alpha_{ij} \in (0,1)$ denote the share of joint production a trait-i agent gets when matched with a trait-j agent.

Assume that the universally unequal sharing norm takes the following form: $\alpha_{ii} = 0.5$ for any i and $\alpha_{ij} > 0.5$ if $i < j$. The society is hierarchical in this case, where traits represent seniority and a more senior agent can get a larger share when matched with a more junior agent. Assume $f(k,k) = \max_i f(i,i)$, which indicates that trait-k is the most productive in terms of own-trait matching. We allow $k \neq 1$ to maintain generality and avoid assuming that the most senior trait is also the most productive in own-trait matching.

For illustration, consider the following examples.

Suppose there are two traits 1 and 2. The payoff structure satisfies that $f(1,1) = 6, f(2,2) = 8$ and $f(1,2) = 10$, and the sharing norm satisfies that $\alpha_{12} = 0.9$ and $\alpha_{21} = 0.1$. In this case, $u_{11} = 3, u_{22} = 4, u_{12} = 9, u_{21} = 1$, which induces PAM, so the evolutionary dynamic is simply driven by the comparison of u_{11} and u_{22}, which leads to a unique stable state $(0,1)$. However, this state is not efficient, because the society can achieve a higher average payoff in the state $(0.5, 0.5)$.

Suppose there are two traits 1 and 2. The payoff structure satisfies that $f(1,1) = 6, f(2,2) = 7$ and $f(1,2) = 10$, and the sharing norm satisfies that $\alpha_{12} = 0.6$ and $\alpha_{21} = 0.4$. In this case, $u_{11} = 3, u_{22} = 3.5, u_{12} = 6, u_{21} = 4$, which induces NAM. To determine the stable state(s), we compare the average payoffs of the trait-1 and trait-2 agents. When $x_1 \leq 0.5$,

$$u_1^*(x) = u_{12} = 6 > F_2(x) = \frac{x_1}{x_2}u_{21} + \frac{x_2 - x_1}{x_2}u_{22} = 4\frac{x_1}{x_2} + 3.5\frac{x_2 - x_1}{x_2}.$$

Hence, no state is stable in this region. When $x_1 > 0.5$,

$$u_1^*(x) = \frac{x_1 - x_2}{x_1}u_{11} + \frac{x_2}{x_1}u_{12} = 3\frac{x_1 - x_2}{x_1} + 6\frac{x_2}{x_1},\ u_2^*(x) = u_{21} = 4.$$

Hence, there exists a unique stable state $(3/4, 1/4)$. However, this state is not efficient, as the society can achieve a higher average payoff in the state $(0.5, 0.5)$.

The examples demonstrate that universally unequal sharing norms can lead to inefficiency through evolution. We have the following general result.

Proposition 3 *For any universally unequal sharing norm, there exists a payoff structure f that leads to an asymptotically stable state that is inefficient.*

4 Evolution of Sharing Norms

In this section, we discuss how sharing norms may evolve. Suppose that on top of the evolution of cultural traits, there is another layer of evolution on sharing norms. Let s denote a sharing norm and let **S** denote the space of sharing norms. There are many populations, and each population is equipped with a different sharing norm. Let $x^*(s)$ denote a stable state in the evolution of traits in the population with sharing norm s and let $X^*(s)$ denote the set of stable states.

Definition 7 *A sharing norm s is* ***evolutionary stable against*** *s' if for every payoff structure f, $\overline{u}^*(x^*(s)) \geq \overline{u}^*((x^*(s'))$ for any $x^*(s) \in X^*(s)$ and $x^*(s') \in X^*(s')$; and for some payoff structure f, $\overline{u}^*(x^*(s)) > \overline{u}^*((x^*(s'))$ for some $x^*(s) \in X^*(s)$ and $x^*(s') \in X^*(s')$.*

We are essentially considering a three-speed dynamic model. At the behavioral level, agents within each population engage in a sequence of matching stages, each of which quickly converges to a stable matching. At the cultural trait level, the distribution of traits evolves relatively more slowly and converges to a stable distribution. At the societal sharing norm level, the sharing norms evolve the slowest.

According to our previous analysis, the equal sharing norm and the market sharing norm are robust in selection on the society level, as they always induce the efficient distribution of traits.

Proposition 4 *The equal sharing norm and the market sharing norm are both evolutionarily stable against any non-equal sharing norms.*

This result rationalizes why the equal sharing norm survives and prevails alongside the market based norm.

5 Conclusions

This paper offers an evolutionary perspective on the resilience of the equal sharing norm and challenges the assumption that market based approaches are inherently superior. Through a two-tier evolutionary framework, we demonstrate how equal sharing, despite its rigidity, supports the evolution of cultural traits that facilitate productive partnerships and thereby enhances long-term productivity. Our findings suggest that the equal sharing norm, far from being an outdated relic, continues to thrive alongside market mechanisms due to its capacity to shape stable and productive populations.

The current model focuses solely on pairwise matching, resulting in a stable distribution of preferences that is either monomorphic or bimorphic. A natural extension would be to consider team formation, and we anticipate that our results would generalize naturally to any fixed team size. It would also be an intriguing direction to explore more general coalition formation settings.

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A Omitted Proofs

Proof of Lemma 1. The set of feasible one-sided matchings is

$$\mathcal{M}_1(x) = \left\{ \mu = (\mu_{ij}) \middle| 2\mu_{ii} + \sum_{j \neq i} \mu_{ij} \leq x_i \, \forall i \in N \text{ and } \mu_{ij} = \mu_{ji} \, \forall i, j \in N \right\}.$$

We define an outcome (μ, w) as the specification of a feasible roommate matching μ and an associated vector of payoffs $(w_i)_{i \in N}$, with w_i denoting the payoff of each individual of trait i. These payoffs have to be feasible: The sum of payoffs across the population has to be equal to the total output under the matching μ. In a one-sided matching μ, the total surplus created is

$$\overline{w}^*(\mu) = \sum_{i \in N} \mu_{ii} f(i,i) + \sum_{i \in N} \left[\sum_{j \in N \setminus \{i\}} \frac{\mu_{ij}}{2} f(i,j) \right] = \sum_{i \in N} \left[\mu_{ii} f(i,i) + \sum_{j \in N \setminus \{i\}} \frac{\mu_{ij}}{2} f(i,j) \right].$$

The maximum of the aggregate surplus over the set of feasible one-sided matchings $\mathcal{M}_1(x)$ is

$$P_1(x) = \max_{\mu \in \mathcal{M}_1(x)} \overline{w}^*(\mu). \tag{P_1}$$

Explicitly, this ***one-sided primal program*** is

$$P_1(x) = \max_{\mu} \ \overline{w}^*(\mu)$$

$$\text{subject to} \qquad 2\mu_{ii} + \sum_{j \in N \setminus \{i\}} \mu_{ij} \leq x_i \quad \forall i \in N, \tag{feasibility}$$

$$\mu_{ij} = \mu_{ji} \quad \forall i, j \in N. \tag{symmetry}$$

Define the following ***one-sided dual program***:

$$D_2(x) = \min_{w = (w_i)_{i \in N}, A = (A_{ij})_{i,j \in N}} \sum_{i \in N} x_i w_i \tag{D_1}$$

$$\text{subject to} \qquad w_i \geq 0 \quad \forall i \in N \tag{individual rationality}$$

$$w_i + w_j \geq f(i,j) + A_{ij} \quad \forall i,j \in N, \qquad \text{(no blocking pair)}$$

$$A_{ij} = -A_{ji} \quad \forall i,j \in N. \qquad \text{(anti-symmetry)}$$

Consider an associated two-sided matching problem, which will be important for our subsequent proofs to leverage the well-established tools in the two-sided matching literature. Denote the set of feasible two-sided matchings by

$$\mathcal{M}_2(x,x) = \left\{ \nu = (\nu_{ij}) \middle| \sum_{j \in N} \nu_{ij} \leq x_i \, \forall i \in N \text{ and } \sum_{i \in N} \nu_{ij} \leq x_j \, \forall j \in N \right\},$$

and the total surplus of a two-sided matching ν by

$$\overline{w}_2^*(\nu) = \sum_{i,j \in N} \nu_{ij} f(i,j).$$

A stable outcome in the two-sided matching setting is defined analogously: (ν, u, v) is a ***two-sided stable outcome*** that consists of a feasible two-sided matching $\nu = (\nu_{ij})_{i,j \in N}$ and two-sided payoff vectors $u = (u_i)_{i \in N}$ and $v = (v_j)_{j \in N}$ if (i) (individual rationality) $u_i \geq 0$ for any $i \in N$ and $v_j \geq 0$ for any $j \in N$, (ii) (no blocking pair) $u_i + v_j \geq f(i,j)$ for any $i, j \in N$, and (iii) (surplus efficiency) $u_i + v_j = f(i,j)$ if $\nu_{ij} > 0$.

The two-sided surplus maximization problem of population state $(x/2, x/2)$—the ***two-sided primal program***—associated with the one-sided primal program (P_1) of population state x is

$$P_2(x/2, x/2) = \max_{\nu \in \mathcal{M}_2(x/2, x/2)} \overline{w}_2^*(\nu). \qquad (P_2)$$

Explicitly,

$$P_2(x/2, x/2) = \max_{\nu} \overline{w}_2^*(\nu)$$

$$\text{subject to} \qquad \sum_{j \in N} \nu_{ij} \leq x_i \forall i \in N \text{ and } \sum_{i \in N} \nu_{ij} \leq x_j \forall j \in N. \qquad \text{(feasibility)}$$

Note that the symmetry constraints in the one-sided primal program are absent in the two-sided primal program. The ***two-sided dual program*** is

$$D_2(x/2, x/2) = \min_{u,v} \sum_{i \in N} \frac{1}{2} x_i u_i + \sum_{j \in N} \frac{1}{2} x_j v_j \tag{D_2}$$

subject to $u_i \geq 0\ \forall i \in N$ and $v_j \geq 0\ \forall j \in N,$ (individual rationality)

$$u_i + v_j \geq f(i, j) \quad \forall i, j \in N. \qquad \text{(no blocking pair)}$$

By Shapley and Shubik (1972) and Gretsky, Ostroy, and Zame (1992), there exist two-sided stable matchings ν and they coincide with the solution of the two-sided primal program (P_2). Moreover, the associated payoffs (u, v) solve the two-sided dual program (D_2). Finally, for any stable matching ν, $\nu_{ij} > 0$ implies $u_i + v_j = f(i, j)$, $\nu_{i0} > 0$ implies $u_i = 0$, and ν_{0j} implies $v_j = 0$.

Note that the two-sided matching problem is a particular case of the one-sided matching problem: If $f(i, j) = -\infty$ whenever i and j belong to the same side, then any optimal or stable matching will be bipartite.

Claim: The values of the one-sided primal program with population state $x \in X_{++}$ and the associated two-sided primal program with population state $(x/2, x/2)$ coincide:

$$P_1(x) = P_2(x/2, x/2).$$

Consider a matching $\mu \in \mathcal{M}_1(x)$. Define ν such that $\nu_{ij} = \mu_{ij}/2$ if $i \neq j$ and $\nu_{ii} = \mu_{ii}$. For any $i \in N$,

$$\sum_{j \in N} \nu_{ij} = \nu_{ii} + \sum_{j \neq i} \nu_{ij} = \mu_{ii} + \sum_{j \neq i} \mu_{ij}/2 \leq x_i/2,$$

and

$$\sum_{j \in N} \nu_{ji} = \nu_{ii} + \sum_{j \neq i} \nu_{ji} = \mu_{ii} + \sum_{j \neq i} \mu_{ji}/2 \leq x_i/2,$$

so $\nu \in \mathcal{M}_2(x/2, x/2)$, as the feasibility constraint for $\nu \in \mathcal{M}_2(x/2, x/2)$ is satisfied.

$$\overline{w}^*(\mu) = \sum_{i \in N} \mu_{ii} f(i,i) + \sum_{i \in N} \left[\sum_{j \in N \setminus \{i\}} \frac{\mu_{ij}}{2} f(i,j) \right] = \sum_{i \in N} \left[\sum_{j \in N} \nu_{ij} f(i,j) \right] = \overline{w}_2^*(\nu).$$

Hence, ν, a feasible two-sided matching in $(x/2, x/2)$, achieves the maximum surplus of one-sided matching under population state x (achieved by the optimal solution μ). We have

$$P_1(x) \leq P_2(x/2, x/2). \tag{1}$$

Conversely, let (ν_{ij}) maximize aggregate surplus over $\mathcal{M}_2(x/2, x/2)$. By the symmetry of f, (ν_{ji}) is also a maximizer. Since (P_2) is a linear program, $\nu'_{ij} = (\nu_{ij} + \nu_{ji})/2$ also maximizes it. Define μ such that $\mu'_{ij} = 2\nu'_{ij}$ if $i \neq j$ and $\mu'_{ii} = \nu'_{ii}$. Then for any $i \in N$,

$$2\mu'_{ii} + \sum_{j \neq i} \mu'_{ij} = 2\nu_{ii} + \sum_{j \neq i} (\nu_{ij} + \nu_{ji}) = \sum_{j \in N} \nu_{ij} + \sum_{j \in N} \nu_{ji} \leq x_i,$$

where $\sum_{j \in N} \nu_{ij} \leq x_i/2$ and $\sum_{j \in N} \nu_{ji} \leq x_i/2$ follow from the feasibility constraints of $\nu \in \mathcal{M}_2(x/2, x/2)$. Hence, $\mu' \in \mathcal{M}_1(x)$. The aggregate surplus under one-sided matching μ' is

$$\sum_{i \in N} \mu'_{ii} f(i,i) + \sum_{i \in N} \left[\sum_{j \neq i} \frac{\mu'_{ij}}{2} f(i,j) \right] = \sum_{i \in N} \nu'_{ii} f(i,i) + \sum_{i \in N} \left[\sum_{j \neq i} \nu'_{ij} f(i,j) \right] = \sum_{i \in N} \sum_{j \in N} \nu'_{ij} f(i,j).$$

Hence, μ', a feasible one-sided matching in population state x, achieves the maximal two-sided surplus in population state $(x/2, x/2)$ (achieved by the optimal solution ν'). Hence,

$$P_1(x) \geq P_2(x/2, x/2). \tag{2}$$

Combining (1) and (2), we have $P_1(x) = P_2(x/2, x/2)$.

By the claim, the values of one-sided primal program (P_1) $P_1(x)$ and two-sided primal program (P_2) $P_2(x/2, x/2)$ coincide. Let $\nu = (\nu_{ij})$ denote a solution of the two-sided primal program. By the duality result of Shapley and Shubik (1972) and Gretsky, Ostroy, and Zame (1992), there exist two-sided stable payoff vectors (u_i) and (v_j) that solve the two-sided dual program (D_2) $D_2(x/2, x/2)$, namely, such that $u_i \geq 0$ for any $i \in N$, $v_j \geq 0$ for any $j \in N$, $u_i + v_j \geq f(i,j)$ for any $i, j \in N$, and $u_i + v_j = f(i,j)$ whenever $\nu_{ij} > 0$.

Define $\mu_{ij} = \nu_{ij} + \nu_{ji}$ when $i \neq j$, $\mu_{ii} = \nu_{ii}$, and $w_i = (u_i + v_i)/2$ for any $i \in N$. We have $w_i \geq 0$, $w_i + w_j \geq f(i,j)$, $w_i + w_j = f(i,j)$ whenever $\mu_{ij} > 0$, and

$$\sum_{i \in N} x_i w_i = \frac{1}{2}\left[\sum_{i \in N} x_i u_i + \sum_{i \in N} x_i v_i\right] = P_2(x/2, x/2) = P_1(x) = \overline{w}^*(\mu).$$

Hence, (μ, w) is a one-sided stable outcome. ■

Proof of Lemma 2. First, fix a population state $x \in X_{++}$. The set of stable one-sided payoffs, $W^*(x)$, is convex. Fix two stable payoff vectors $w^{\spadesuit} \in W^*(x)$ and $w^{\diamondsuit} \in W^*(x)$. Take any $\lambda \in (0,1)$. We have (i) $\lambda w_i^{\spadesuit} + (1-\lambda) w_i^{\diamondsuit} \geq 0$ for any $i \in N$, because $w_i^{\spadesuit} \geq 0$ and $w_i^{\diamondsuit} \geq 0$ for any $i \in N$, (ii) $[\lambda w_i^{\spadesuit} + (1-\lambda) w_i^{\diamondsuit}] + [\lambda w_j^{\spadesuit} + (1-\lambda) w_j^{\diamondsuit}] \geq f(i,j)$ for any $i, j \in N$, because $w_i^{\spadesuit} + w_j^{\spadesuit} \geq f(i,j)$ and $w_i^{\diamondsuit} + w_j^{\diamondsuit} \geq f(i,j) \geq f(i,j)$.

Second, fix a population state $x \in X_{++}$. The set of stable one-sided payoffs, $W^*(x)$, is compact. Since $W^*(x)$ is a set in $\mathbb{R}^{|I|}$, it suffices to show that $W^*(x)$ is closed and bounded. Each element of $W^*(x)$ is bounded due to the surplus function being bounded. The set is also closed, because all individual rationality and no-blocking-pair constraints are non-strict inequalities.

Third, the correspondence, $W : X_{++} \rightrightarrows \mathbb{R}^{|I|}$, that specifies the set of stable one-sided payoffs $W^*(x)$ for population state $x \in X_{++}$, is upper-hemicontinuous. Proof: $W : X_{++} \rightrightarrows \mathbb{R}^{|I|}$ is upper-hemicontinuous if and only if $x^k \to x$, $w^k \in W(x^k)$, and $w^k \to w$ imply $w \in W^*(x)$. It suffices to show that $\overline{w}^*(x) = \sum_{i \in N} w_i x_i$ and that w satisfies $w_i \geq 0$ for any $i \in N$ and $w_i + w_j \geq f(i,j)$ for any $i, j \in N$. First, we show that $\overline{w}^*(x) = \sum_{i \in N} w_i x_i$.

$$\left|\sum_{i \in N} w_i x_i - \overline{w}^*(x)\right|$$

$$\begin{aligned}
&= \left|\sum_{i\in N} w_i x_i - \sum_{i\in N} w_i^k x_i^k + \overline{w}^*(x^k) - \overline{w}^*(x)\right| \\
&\leq \left|\sum_{i\in N} w_i x_i - \sum_{i\in N} w_i^k x_i^k\right| + \left|\overline{w}^*(x^k) - \overline{w}^*(x)\right| \\
&\leq \left|\sum_{i\in N} w_i x_i - \sum_{i\in N} w_i x_i^k\right| + \left|\sum_{i\in N} w_i x_i^k - \sum_{i\in N} w_i^k x_i^k\right| + \left|\overline{w}^*(x^k) - \overline{w}^*(x)\right| \\
&\leq \left|\sum_{i\in N} w_i x_i - \sum_{i\in N} w_i x_i^k\right| + \left|\left|w - w^k\right|\right| + \left|\overline{w}^*(x^k) - \overline{w}^*(x)\right|.
\end{aligned}$$

Fix ϵ. Since $x^k \to x$, there is a $k_1(\epsilon)$ such that all $k > k_1(\epsilon)$, $\left|\sum_{i\in N} w_i x_i - \sum_{i\in N} w_i x_i^k\right| \leq \epsilon/3$. Since $w^k \to w$, there is a $k_2(\epsilon)$ such that for all $k > k_2(\epsilon)$, $\left|\left|w - w^k\right|\right| \leq \epsilon/3$. Since $x^k \to x$, $\left|\overline{w}^*(x^k) - \overline{w}^*(x)\right| \leq \epsilon/3$. Therefore, $\left|\sum_{i\in N} w_i x_i - \overline{w}^*(x)\right| \leq \epsilon$ for any $\epsilon > 0$.

We also need to show that w satisfies the stability conditions. Suppose otherwise: $w_i + w_j < f(i,j)$ for a positive measure of (i,j) or $w_i < 0$ for some $i \in N$. Then $\lim_{k\to\infty} \|w - w^k\| > 0$, a contradiction to the assumption that $w^k \to w$. ■

In what follows, we provide the theorems we will depend on to use the Lyapunov method to show convergence and stability of the dynamics.

Theorem 2 *(Sandholm, 2010, Theorem 7.B.2) Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of A. Let $L : Y \to R+$ be Lipschitz continuous with $L^{-1}(0) = A$. If each solution $\{x^t\}_{t\geq 0}$ of $\dot{x} \in V^{RDI}(x)$ (or $\dot{x} = V^{RDE}(x)$) satisfies $\dot{L}(x^t) \leq 0$ for almost all $t \geq 0$, then A is Lyapunov stable.*

Theorem 2 states that the existence of a Lipschitz continuous Lyapunov function $L(x)$ guarantees that $L^{-1}(0)$, the set of population states that minimize $L(x)$, is Lyapunov stable.

Theorem 3 *(Sandholm, 2010, Theorem 7.B.4) Let $Y \subset X$ be relatively open and inescapable under (RDI). Let $L : Y \to R$ be C^1 and satisfy (i) $\frac{\partial L}{\partial v}(x) \equiv \nabla L(x)'v \leq 0$ for all $v \in V^{RDI}(x)$ and $x \in Y$ and (ii) $\left[\mathbf{0} \notin V^{RDI}(x) \implies \frac{\partial L}{\partial v}(x) < 0\right]$ for all $v \in V^{RDI}(x)$ and $x \in Y$. Then, for all solutions $\{x^t\}$ of (RDI) with $x_0 \in Y$, $\omega(\{x^t\}) \subseteq \{x \in Y : \mathbf{0} \in V(x)\}$.*

Theorem 3 shows that the existence of a C^1 Lyapunov function $L(x)$ guarantees the solution trajectories converge to the states that minimize $L(x)$. Combining Theorems 2 and 3 gives us globally asymptotic stability of the set of population states that minimize $L(x)$ under (RDI).

Proof of Proposition 1. In the first case, $L^{RDI}(x) = -\log x_i$ is a C^1 function. Moreover, for any x such that $x_i > 0$, and for any $v = x \cdot [w^*(x) - \overline{w}^*(x)]$, we have

$$\begin{aligned} \frac{\partial L^{RDI}}{\partial v}(x) &= \nabla L^{RDI}(x)'v \\ &= -\left(\frac{1}{x_i}\right) x_i[w_i^*(x) - \overline{w}^*(x)] \\ &= -[w_i^*(x) - \overline{w}^*(x)]. \end{aligned}$$

By the definition of stable matching, $w_i^*(x) \geq f(i,i)/2$. By the fact that $f(i,i) = \max_{k,l} f(k,l)$, $f(i,i)/2 > \overline{w}^*(x)$ as long as $x \neq e_i$. Therefore, we have $\frac{\partial L^{RDI}}{\partial v}(x) < 0$. At $x^* = e_i$, the stable matching is given by $\mu_{ii}(e_i) = 0.5$, where $w_i^*(e_i) = \overline{w}^*(e_i) = f(1,1)/2$. Hence, $V^{RDI}(e^1) = \{\mathbf{0}\}$. Also, $L^{-1}(0) = \{e_i\}$. By Theorems 2 and 3, e_i is the globally asymptotically stable state with respect to $X \backslash \{x | x_i = 0\}$. In addition, $f(i,i) = \max_{k,l} f(k,l)$ implies that e_i is the unique efficient state.

In the second case, $L^{RDI}(x) = -[\log(2x_i) + \log(2x_j)]$ is a C^1 function. Moreover, for any x such that $x_i > 0$ and $x_j > 0$, and for any $v = x \cdot [w^*(x) - \overline{w}^*(x)]$, we have

$$\begin{aligned} \frac{\partial L^{RDI}}{\partial v}(x) &= \nabla L^{RDI}(x)'v \\ &= -\left(\frac{1}{x_i}\right) x_i[w_i^*(x) - \overline{w}^*(x)] - \left(\frac{1}{x_j}\right) x_j[w_j^*(x) - \overline{w}^*(x)] \\ &= -[(w_i^*(x) + w_j^*(x)) - 2\overline{w}^*(x)]. \end{aligned}$$

By the definition of stable matching, $w_i^*(x) + w_j^*(x) \geq f(i,j)$. By the fact that $f(i,j) = \max_{k,l} f(k,l)$, we have $f(i,j)/2 > \overline{w}^*(x)$. Therefore, we have $\frac{\partial L^{RDI}}{\partial v}(x) < 0$. At x^* such that $x_i^* = x_j^* = 0.5$, the stable matching is given by $\mu_{ij}(x^*) = 0.5$, where $w^*(x^*)$ that satisfies $w_i^*(x^*) = w_j^*(x^*) = f(i,j)/2$ is a possible stable payoff vector. Hence $\mathbf{0} \in V^{RDI}(x^*)$. Also,

$L^{-1}(0) = \{x^*\}$. By Theorems 2 and 3, x^* is the globally asymptotically stable state with respect to $X\backslash\{x|x_i = 0 \text{ or } x_j = 0 \text{ or both}\}$. In addition, $f(i,j) = \max_{k,l} f(k,l)$ implies that x^* is the unique efficient state. ■

Theorem 4 *(Sandholm, 2010, Corollary 7.B.6) Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of A. Let $L : Y \to R$ be C^1 with $L^{-1}(0) = A$. If $\dot{L}(x) < 0$ for all $x \in Y - A$, then A is asymptotically stable under $\dot{x} = V^{RDE}(x)$. If in addition, $Y = X$, then A is globally asymptotically stable under $\dot{x} = V^{RDE}(x)$.*

Theorem 4 gives us globally asymptotic stability of the set of population states that minimize $L(x)$ under (RDE).

Proof of Proposition 2. In the first case, $f(1,1) = \max_{k,l} f(k,l)$ implies that $f(1,1) > f(1,i)$ for any $i \neq 1$. Hence, as long as $x_1 > 0$, all i-trait agents must match among themselves and $u_1^*(x) = u_{11} = f(1,1)/2$ is the highest payoff an agent can obtain in the population.

We want to show that $L^{RDE}(x) = -\log x_1$ serves as a Lyapunov function for the replicator dynamic. First, it is a C^1 function with $L^{RDE}(x^*) = 0$. Furthermore, for any x^t such that $x_1^t > 0$,

$$\begin{aligned} dL^{RDE}(x^t)/dt &= \bigtriangledown L^{RDE}(x^t)'\dot{x}^t \\ &= -\left(\frac{1}{x_1^t}\right) x_1^t[u_1^*(x^t) - \overline{u}^*(x^t)] \\ &= \overline{u}^*(x^t) - u_1^*(x^t) \\ &< 0 \text{ as long as } x^t \neq e_1. \end{aligned}$$

Hence, according to Theorem 4, x^* is globally asymptotically stable with respect to $X\backslash\{x|x_1 = 0\}$.

In the second case, $f(i,j) = \max_{k,l} f(k,l)$ implies that $f(i,j) > f(i,i)$ and $f(i,j) > f(j,j)$. Hence, these two traits of agents will match with each other to the greatest extent possible. Without loss of generality, assume that $0 < x_i^0 < x_j^0$. Then all trait-i agents are matched with trait-j agents, while the remaining trait-j agents match otherwise. In this case, $u_i^*(x^0) = u_{ij} = f(i,j)/2$, which is the largest payoff an agent can get.

Let $r(t) = x_i^t/x_j^t$. Observe that under the replicator dynamic, $\frac{dr(t)}{dt} = \frac{x_i^t}{x_j^t}[u_i^*(x^t) - u_j^*(x^t)] > 0$ as long as $x_i^t < x_j^t$ so that $u_i^*(x^t) - u_j^*(x^t) > 0$. Also, once $x_i^t = x_j^t$, $r(t) = 1$ and $u_i^*(x^t) = u_j^*(x^t) = f(i,j)/2$ because all trait-i agents are exactly matched with all trait-j agents. Hence, $\frac{dr(t)}{dt} = 0$. In sum, $r(t)$ is a strictly increasing function bounded by 1, implying that $\lim_{t\to\infty} r(t) = 1$. Define $Y = \{x|x_i = x_j > 0\}$. This part of the proof shows that the dynamic always converges to Y from $X\backslash\{x|x_i = 0 \text{ or } x_j = 0 \text{ or both}\}$.

Next, we want to show that x^* is asymptotically stable with respect to Y. $L^{RDE}(x) = -[\log(2x_j) + \log(2x_i)]$ is a C_1 function in Y with $L^{RDE}(x^*) = 0$. In addition,

$$\begin{aligned}
dL^{RDE}(x^t)/dt &= \bigtriangledown L^{RDE}(x^t)'\dot{x}^t \\
&= -\left(\frac{1}{2x_i^t}\right) x_i^t \left[u_i^*(x^t) - \overline{u}^*(x^t)\right] - \left(\frac{1}{2x_j^t}\right) x_j^t \left[u_j^*(x^t) - \overline{u}^*(x^t)\right] \\
&= \overline{u}^*(x^t) - \left[u_i^*(x^t) + u_j^*(x^t)\right]/2 \\
&= \overline{u}^*(x^t) - f(i,j)/2 \\
&< 0 \text{ as long as } x^t \neq x^*.
\end{aligned}$$

Hence, $L^{RDE}(x)$ is a Lyapunov function for the replicator dynamic on Y and once the dynamic reaches Y, the dynamic converges to x^* according to Theorem 4. In sum, we can conclude that x^* is globally asymptotically stable with respect to $X\backslash\{x|x_i = 0 \text{ or } x_j = 0 \text{ or both}\}$. ■

Proof of Proposition 3. We will prove by construction. First, let $f(i,i) = c$ for some constant $c > 0$, for any $i \neq 1$. For any $1 < i < j$, let $f(i,j) = c + \nu_{ij}$, where $0 < \nu_{ij} < c[1/(2\alpha_{ji}) - 1]$. We must have that for any $1 < i < j$,

$$u_{ji} = \alpha_{ji}(c + \nu_{ij}) < u_{jj} = u_{ii} = c/2 < u_{ij} = \alpha_{ij}(c + \nu_{ij}).$$

This implies that in any stable matching, no $i - j$ match will be formed.

Next, let $f(1,i) = c + \epsilon_i$ for any $i \neq 1$, where $0 < \epsilon_i < c[1/(2\alpha_{i1}) - 1]$. Let $\epsilon = \min_i \epsilon_i$ and let $f(1,1) = c + \epsilon/2$. Then we have, for any $i \neq 1$,

$$u_{i1} = \alpha_{i1}(c + \epsilon_i) < c/2 = u_{ii} < u_{11} = (c + \epsilon/2)/2 < (c + \epsilon_i)/2 < \alpha_{1i}(c + \epsilon_i) = u_{1i}.$$

This implies that in any stable matching, no $1 - i$ match will be formed. Hence, the unique stable matching must be PAM and e_1 is the globally asymptotically stable state $X \backslash \{x | x_1 = 0\}$. However, it is not efficient because $f(1,1) < f(1,i)$ for any $i \neq 1$. ■

Proof of Proposition 4. It is directly implied by Propositions 1, 2 and 3. ■