ECON201 Midterm Review

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This is not meant to be a comprehensive review for the upcoming midterm, but rather some sporadic extended remarks on the common problems from the problem sets noticed by the TA.

1 Cost Functions

Some basic definitions that you should be familiar with:

- **Total Cost Function** $C(q)$ is the total cost of producing q units. It may include fixed cost F of setting up the plant and variable cost $VC(q)$ that varies by quantity. $C(q) = F + VC(q)$.
- **Average Cost Function** $C(q)/q$ is the per-unit cost of producing q units. **Average Variable Cost Function** is $AVC(q) = VC(q)/q$.
- **Marginal Cost Function** $MC(q) = dC(q)/dq = C'(q)$ is the cost of producing the q^{th} unit.

2 Homogenous Production Function

In general, a homogenous function $f(\mathbf{x})$ of degree k is defined as a function such that for $t > 0$, $f(t\mathbf{x}) = t^k f(\mathbf{x})$. For example, $f(x,y) = x^2 + y^2$ is homogenous of degree 2, as $f(tx,ty) = (tx)^2 + (ty)^2 = t^2x^2 + t^2y^2 = t^2 f(x,y)$. Some special functions of homogenous of degree **one** are CRS production functions, Cobb-Douglas production function $f(x,y) = x^{\alpha}y^{1-\alpha}$, and linear function $f(x) = cx$; however, homogenous production function does not imply that the function is homogenous of degree one.

Now suppose that there are inputs x_i that produce output q, according to the homogenous production function of degree k, $q = f(x_1, \cdots, x_n)$. The cost function $c(q)$ denotes the total cost of producing q, which is $\sum_i p_i x_i$. Now what is the cost of producing rq? The inputs that are needed to produce rq is

$$rq = \left(r^{1/k}\right)^k q = \left(r^{1/k}\right)^k f(x_1, \cdots, x_n) = f\left(r^{1/k}x_1, \cdots, r^{1/k}x_n\right)$$

where the last equality comes from that the production function is homogenous of degree k. Therefore, the cost function of producing rq is

$$c(rq) = \sum p_i r^{1/k} x_i = r^{1/k} \sum p_i x_i = r^{1/k} c(q).$$

That is, when the production function is homogenous of degree k, the cost function is homogenous of degree $1/k$.

Exercise (PS2, TFU4). If all firms have homogeneous production technology and larger firms are homogeneous of a higher degree, then policies aimed at promoting small business are well-justified.

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Solution. For a firm with homogenous production function, the cost is $c(q) = kq^{1/k}$, in equilibrium $p = MC(q) = c'(q) = q^{\frac{1-k}{k}}$, so $q \equiv S(p) = p^{\frac{k}{1-k}}$. The profit function is

$$\pi(p) = \text{rev} - \text{cost} = p \cdot S(p) - c(S(p)) = pp^{\frac{k}{1-k}} - k\left(p^{\frac{k}{1-k}}\right)^{\frac{1}{k}} = (1-k)\, p^{\frac{1}{1-k}}.$$

The fraction of revenue kept as profit is

$$\frac{\pi(p)}{\text{rev}(p)} = \frac{(1-k)\, p^{1/(1-k)}}{p \cdot p^{k/(1-k)}} = 1-k,$$

which decreases as $k\,(<1)$ increases. Therefore, the smaller businesses, despite earning less, keep more fraction of their revenues as profits, thus paying less to the workers; consequently, policies aimed at promoting small businesses are not well-justified.

Exercise. 2011PS2, #2; PS1, TFU4; PS1, SA2.

3 Lagrange Multiplier and Its Economic Interpretation

In general, a constrained optimization problem is

$$\begin{aligned} & \max_x f(\mathbf{x}) \\ \text{s.t.} \quad & g(\mathbf{x}) \le c \end{aligned}$$

The Lagrangian is then

$$\mathscr{L} = f(\mathbf{x}) + \lambda\,(c - g(\mathbf{x}))$$

where λ is the **Lagrange multiplier** which is non-negative. By Envelope Theorem, $\lambda = \partial \mathscr{L}/\partial c$, which is interpreted as the marginal increase when c, the constraint increases. Equivalently, that is saying, when c decreases, the feasible set of x becomes smaller, so there is some opportunity cost associated with the contracted constraint (feasible) set of x. in other words, λ is the opportunity cost of having the smaller constraint set rather than the bigger one. To take a concrete example. Consider the standard problem you have encountered with the utility maximization subject to fixed income.

Example. Suppose that utility function is $u(x,y) = x^{\alpha} y^{1-\alpha}$, and prices of the goods are p_x and p_y, with income w, then the utility maximization problem is

$$\max_{x,y} u(x,y) = x^{\alpha} y^{1-\alpha} \quad \text{s.t. } p_x x + p_y y \le w.$$

Then the Lagrangian is

$$\mathscr{L} = x^{\alpha} y^{1-\alpha} + \lambda\,(w - p_x x - p_y y).$$

The first order conditions (FOCs) are

$$\begin{aligned} [x]: \quad \alpha x^{\alpha-1} y^{1-\alpha} - \lambda p_x &= 0 \\ [y]: \quad (1-\alpha)\, x^{\alpha} y^{1-\alpha} - \lambda p_y &= 0 \\ [\lambda]: \quad p_x x + p_y y &= w \end{aligned}$$

Or simplify,

$$\begin{aligned} \alpha\,(x/y)^{\alpha-1} &= \lambda p_x \\ (1-\alpha)\,(x/y)^{\alpha} &= \lambda p_y \end{aligned}$$

Divide each other, we get

$$\frac{1-\alpha}{\alpha} = \frac{p_y y}{p_x x},$$

or $x = \alpha w/p_x$ and $y = (1-\alpha)\, w/p_y$. To get λ, we plug x and y into the equation above,

$$\lambda = \alpha \left(\frac{p_y}{p_x} \frac{\alpha}{1-\alpha} \right)^{\alpha-1} / p_x = \left(\frac{\alpha}{p_x} \right)^{\alpha} \left(\frac{1-\alpha}{p_y} \right)^{1-\alpha}.$$

Note that the utility obtained by consuming the optimally chosen bundle is

$$u(p_x, p_y, w) = \left(\frac{\alpha}{p_x} \right)^{\alpha} \left(\frac{1-\alpha}{p_y} \right)^{1-\alpha} w = \lambda w.$$

This expression is true for any w, given p_x and p_y. Therefore, for any unit increase in income w, there is corresponding λ unit increase in utility; in other words, $\lambda = \partial \mathscr{L}/\partial w = \partial u/\partial w$ is the marginal utility of income.

λ is indeed the marginal utility of income, or in general, the opportunity cost of having the constraint.

Exercise (PS1, LA2c)**.** You are a monopolist facing the inverse demand curve $P(q) = 1/\sqrt{q}$. You have no marginal costs, but you cannot produce more than 5 units because of constraints on your capacity.

Solution. The problem is $\max_q P(q)\, q$ such that $q \leq Q$. The Lagrangian is

$$\mathscr{L} = R(q) + \lambda (Q - q) = P(q)\, q + \lambda (Q - q).$$

Since there is no marginal cost, there is only quantity constraint by Q, with $q^* = Q$.

$$\mathscr{L} = \frac{1}{\sqrt{Q}} Q = \sqrt{Q}.$$

Then the Lagrangian multiplier is

$$\lambda = \frac{\partial R(Q)}{\partial Q} = \frac{\partial \mathscr{L}}{\partial Q} = \frac{1}{2\sqrt{Q}}.$$

For $Q = 5$, the Lagrange multiplier is $1/\left(2\sqrt{5}\right)$.

Hopefully by now you are half convinced that the Langrangian multiplier represents the opportunity cost of a constraint. Let's try to look at a problem without setting up the problem.

Exercise (PS1, LA2b)**.** You have five special occasions planned for the year. You have three bottles of champagne. The value to you of having the bottle to drink on the different occasions is independent across occasions and is, in chronological order, \$10, \$50, \$13.50, \$20, \$200.

Solution. The marginal value of having the 3^{rd} unit is \$20 and the marginal value of having 4^{th} \$13.50. λ is the opportunity cost of the constraint, so it's bounded by the marginal value of third and fourth units: $13.5 \leq \lambda \leq 20$.

Exercise. 2011PS1, Problem 3; PS1, LA2a-e

4 Power Law and Superstar Effect

Talent is extremely fat-tailed, and it follows the **Power Law**,

$$F(x;\alpha,x_{\min}) \equiv \Pr(X<x) = 1-\left(\frac{x}{x_{\min}}\right)^{-\alpha}.$$

In particular, when $\alpha = 1$, it is called the **Zipf's Law**.

The consequence of the **superstar effect**: In the long run, it might not reach the competitive level and free entry; instead, rents might increase because of the talent required in the industry.

Exercise (PS2, LA2d)**.** Suppose that salaries follow a power law within profession with coefficient $\alpha = 3$. Is it better to the 20$^{\text{th}}$ best professional in a field where the highest paid professional earns five times as much as in another field or to be the highest paid individual in that other field? How about if $\alpha = 1$?

Solution. The n^{th} best person in a profession earns $n^{-1/\alpha}$ fraction of the highest paid person. Therefore, the 20th best professional earns $20^{-1/3} \approx 0.37$ of the highest professional in the field, but it is more than $1/5 = 0.2$ of the highest professional, if to be the highest in the other field. However, if $\alpha = 1$, $20^{-1} = 0.05$, the order of preference is flipped.

A good read on the Power Laws and their applications is Gabaix [2009] (final version available on Chalk).

5 Elasticities - Storables vs. Durables, LR vs. SR

Given a function $f(x,y)$, in general, x-elasticity of f is defined as the percentage change of f due to percentage change of x. Mathematically, the following expressions are all equivalent:

$$\varepsilon_f = \frac{d\ln f(x,y)}{d\ln x} = \frac{df(x,y)/f(x,y)}{dx/x} = \frac{df(x,y)}{dx}\cdot\frac{x}{f(x,y)}.$$

The most common elasticity we think about is the price elasticity of demand: given the demand function $Q(P)$, the price elasticity of demand is

$$\varepsilon_D = \frac{dQ(P)}{dP}\cdot\frac{P}{Q(P)}.$$

Example (Problem Set 2, LA2a,b)**.** Suppose the world lasts for only four years, there is no discounting and the demand for oil each year i is given by $Q_i(p_i) = \delta_i p_i^{-\varepsilon}$, where $\delta_i, \varepsilon > 0$. Suppose all δ_i are initially equal to 1 and that there is enough oil to supply for 4 years at one unit of oil a year (or two years of two units with no units the other two years, etc.) and that oil has no marginal cost of supply.

1. if δ_i rises by one percent
   (a) By what percent does the price in year i rise?
   (b) By what percent does the quantity of oil rise in year i?
   (c) What is the short-run elasticity of supply?
2. if all δ_i rise by one percent
   (a) How much does price rise?

(b) How much does quantity in year i rise?

(c) What is the long-term elasticity of supply?

3. How does this relate to the nature of good that oil is?

Solution. The key is to notice that the price is constant through all periods because there is no discounting (if there is discounting r, then the prices are $p/(1+r)^{t-1}$). So the total quantity demanded is

$$Q^d = \sum_i Q_i(p) = \sum_i \delta_i p^{-\varepsilon} = p^{-\varepsilon} \sum_i \delta_i.$$

In equilibrium, this equals $Q^s = 4$, so $p^{-\varepsilon} = 4/\sum_i \delta_i$. Take natural log of both sides,

$$-\varepsilon \ln p = \ln 4 - \ln \sum_i \delta_i,$$

and take the derivative on both sides,

$$\varepsilon \frac{dp}{p} = \frac{\sum_i d\delta_i}{\sum_i \delta_i}.$$

Suppose that all $\delta_i = 1$, then

$$\frac{dp}{p} = \frac{1}{4\varepsilon} \sum_i \frac{d\delta_i}{1}. \tag{1}$$

The δ_i-elasticity of quanity is

$$\frac{\partial \ln Q_i(\delta_i, p)}{\partial \ln \delta_i} = \frac{\partial Q_i(\delta_i, p)}{\partial \delta_i} \frac{\delta_i}{Q_i(\delta_i, p)} = p^{-\varepsilon} \frac{\delta_i}{\delta_i p^{-\varepsilon}} = 1. \tag{2}$$

1. If $d\delta_i = 1\%$,

   (a) $d\ln p = (1/(4\varepsilon))\%$.

   (b) $d\ln Q_i = 1 \cdot d\ln \delta_i = 1\%$.

   (c) Now the δ_i elasticity of quantity demanded (supplied, equating each other),

$$\begin{aligned}
& \frac{dQ_i(\delta_i, p)}{d\delta_i} \cdot \frac{\delta_i}{Q_i(\delta_i, p)} \\
= \; & \frac{\partial Q_i(\delta_i, p)}{\partial \delta_i} \cdot \frac{\delta_i}{Q_i(\delta_i, p)} + \frac{\partial Q_i(\delta_i, p)}{\partial p} \cdot \frac{\partial p}{\partial \delta_i} \cdot \frac{\delta_i}{Q(\delta_i, p)} \\
= \; & \frac{\partial Q_i(\delta_i, p)}{\partial \delta_i} \frac{\delta_i}{Q_i(\delta_i, p)} + \left[\frac{\partial Q_i(\delta_i, p)}{\partial p} \frac{p}{Q_i(\delta_i, p)}\right] \left[\frac{\partial p}{\partial \delta_i} \frac{\delta_i}{p}\right] Q_i(\delta_i, p) \frac{\delta_i}{Q_i(\delta_i, p)} \\
= \; & \frac{\partial \ln Q_i(\delta_i, p)}{\partial \ln \delta_i} + \frac{\partial \ln Q_i(\delta_i, p)}{\partial \ln p} \cdot \frac{\partial \ln p}{\partial \ln \delta_i} \delta_i \qquad (3) \\
= \; & 1 + (-\varepsilon) \frac{1}{4\varepsilon} = \frac{3}{4},
\end{aligned}$$

   where

$$\frac{d\ln Q_i(\delta_i, p)}{d\ln p} = \delta_i (-\varepsilon) p^{-\varepsilon - 1} \frac{p}{\delta_i p^{-\varepsilon}} = -\varepsilon.$$

   Therefore,

$$\frac{d\ln Q_i(\delta_i, p)}{d\ln p} = \frac{d\ln Q_i(\delta_i, p)}{d\ln \delta_i} / \frac{d\ln p}{d\ln \delta_i} = \frac{3}{4} / \frac{1}{4\varepsilon} = 3\varepsilon.$$

2. If $d\delta_i = 1\%$ for all $i = 1, \cdots, 4$,

   (a) By equation 1, $d\ln p = 4\%/(4\varepsilon) = (1/\varepsilon)\%$.

   (b) By equations 2 and 3,
   $$\frac{d\ln Q_i(\delta_i, p_i)}{d\ln \delta_i} = 1 + (-\varepsilon)\frac{1}{\varepsilon} = 0.$$

   (c) The long-run elasticity is $d\ln Q_i / d\ln p = 0$.

Therefore, it is much more elastic in the SR than in the LR: it is because that oil is perfectly storable. In general, **storables are more elastic in the short run and durables are more elastic in the long run**. The producer surplus (PS) is greater for more inelastic supply curves (Figure 1). Therefore, storables have higher PS in the LR than the SR (lower PS in the SR), but durables have higher PS in the SR than the LR (lower PS in the LR than SR). To distinguish storables and durables, read class notes Durables and Storable Factors.

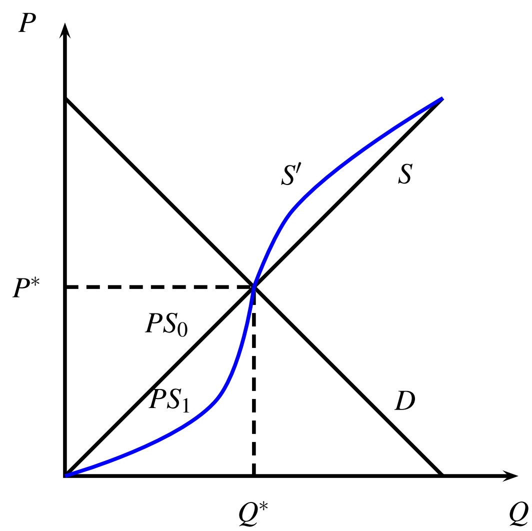

Figure 1: Producer Surplus is Greater with Inelastic Supply Curve

Exercise. PS2: TFU1, TFU5, SA1, LA1a, b, LA2a-c.

6 Pecuniary and Real Externalities

Let's go to Wikipedia for a fairly accurate description: "A **pecuniary externality** is an externality which operates through prices rather than through real resource effects... This is in contrast with **real externalities** which have a direct resource effect on a third party."

Example (PS3, LA2a)**.** A generic drug enters to compete with a branded monopolist. Prior to the entry, the incumbent had a price of \$1000 for a year's treatment and sold a million units a year. After the entry the price of the incumbent falls to \$100 for a year's treatment and her sales fall to half a million units. Assume neither the monopolist nor the entrant have any marginal costs of production.

Solution. The monopolist's profit before the entrant is $10^6 \times \$10^3$ and the profit after is $\frac{1}{2} \times 10^6 \times \$10^2$. The change in the monopolist's profit is purely because of the entrant and the entrant is not paying anything to the monopolist, so the change in profit is all externalities (shaded region). Whether they are real or pecuniary, we need to consider whether the consumers are getting the benefits as a result of monopolist's loss. Let's demonstrate with a demand curve and zero supply curve (as assumed in the question).

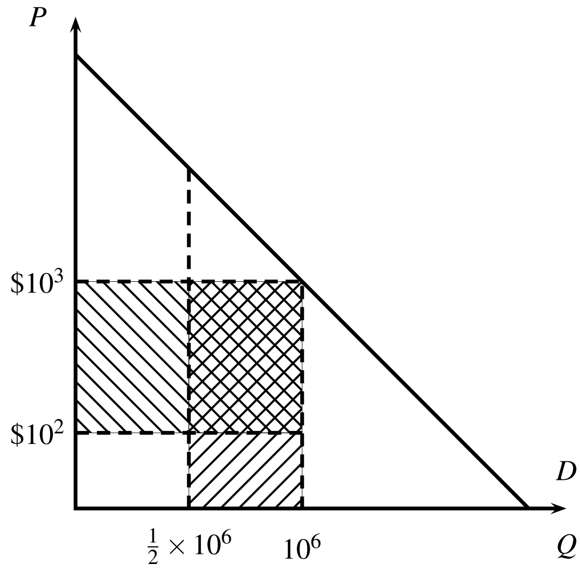

Figure 2: Real and Pecuniary Externalities

The real externalities to the monopolist are the reduction of half a million in sales. Each unit is worth between \$100 and \$1000, so the total real externality is between $\frac{1}{2} \times 10^6 \times \$10^2$ and $10^6 \times \$10^2$. Whether the crosshatched area is real or pecuniary externality depends on how much the entrant sells them at what price; if it becomes consumer surplus for the additional unit sold, then it is pecuniary externality. If it becomes the entrant's profit, then it's real externality for the monopolist.

On the other hand, for the units that the original monopolist is still selling, the reduced profits are transferred to the consumers in the form of consumer surplus, so they are pecuniary externalities. The bound on the pecuniary externality is then right-stripped (?) area: $\left(\frac{1}{2} \times 10^6 \times \$900, 10^6 \times \$900\right)$.

Exercise. PS3, LA2a-e.

7 Cap-and-Trade vs. Pigouvian Tax

We see that in many settings, there are externalities, regardless the classifications as real and/or pecuniary. The government needs to step in in these settings to correct the wrong, i.e. to internalize the externalities. The primary example is toy factories polluting a river nearby. The children benefit from the toy, given by the demand (marginal private value) curve. The toy factories has a supply (marginal cost) curve that does not depend on the cost of the pollution to the river. The government counts both the cost of the toy factory as well as the cost to the pollution as social cost.

The government wants the quantity produced to be q^{**} such that marginal benefit equates marginal social cost; however, the factories want to produce $q^* > q^{**}$. In order to let the factories to produce q^{**}, the government has two possible policies. The first is **cap-and-trade**: to impose a total cap (quota) $\bar{q} = q^{**}$, which can be equally allocated to the toy factories, and allow them to trade the capacities freely. Then if the two factories differ by size and productivity, they will make beneficial trades. The second is **Pigouvian tax**: to impose a per-quantity tax t such that marginal cost plus tax equals marginal social cost; in essence, the factory is paying for the cost from pollution; the externalities are internalized. The factories will optimize according to their new marginal cost curve. The two policies will have the same outcome: the same quantity produced and the same equilibrium price.

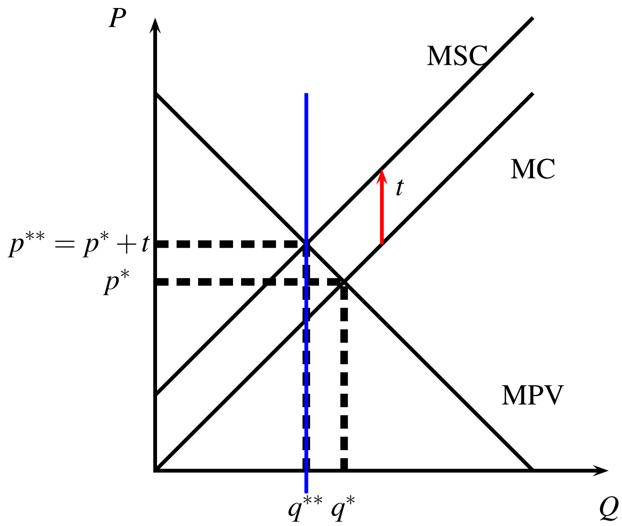

Figure 3: Cap-and-Trade vs. Pigouvian Tax in a Figure

However, this is under idealized setting where there is no uncertainty about the demand at all. The Price vs. Quantities note, which is a stripped down version of Weitzman [1974], shows that the two policies differ when there are linear demand and cost functions but uncertainty in demand.

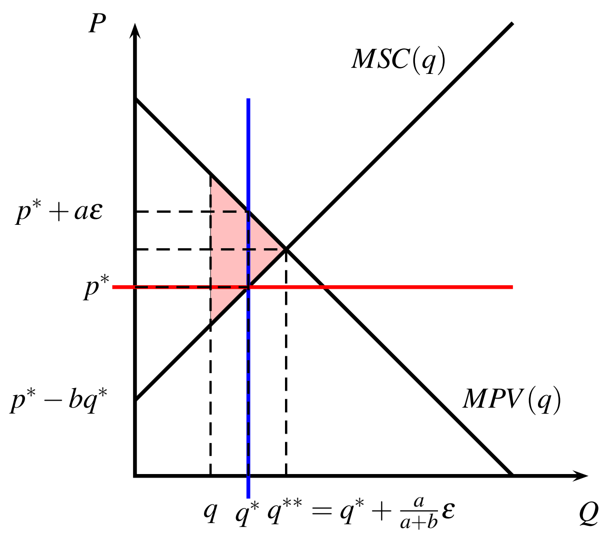

Figure 4: Cap-and-Trade vs. Pigouvian Tax under Uncertain Demand with $\varepsilon > 0$

$$\begin{aligned}
MPV(q) &= p^* - a(q - q^* - \varepsilon) \\
MSC(q) &= p^* + b(q - q^*)
\end{aligned}$$

In general, for any q, the difference between MPV and MSC if ε is known:

$$\Delta(q) = MPV(q) - MSC(q) = -(a+b)(q-q^*) + a\varepsilon.$$

If ε is known, the socially optimal allocation, or the optimal cap, is to set at $\Delta(q) = 0$,

$$-(a+b)(q^{**}-q^*) + a\varepsilon = 0 \Rightarrow q^{**} = q^* + \frac{a}{a+b}\varepsilon.$$

Plugging in q^{**} into Δ,

$$\Delta(q) = -(a+b)\left(q - q^{**} + \frac{a}{a+b}\varepsilon\right) + a\varepsilon = (a+b)(q^{**}-q).$$

The **Harberger Triangle** associated with choosing q instead of q^{**} is $\left|\frac{1}{2}\Delta(q)\cdot(q^{**}-q)\right| = \frac{1}{2}(a+b)(q^{**}-q)^2$ (demonstrated by pink).

To minimize the expected loss over all possible states, we choose q to

$$\min_q \mathbb{E}\left[\frac{1}{2}(a+b)(q^{**}-q)^2\right].$$

This is equivalent to choose q to minimize $\mathbb{E}\left[(q^{**}-q)^2\right]$ as a and b are known. FOC is

$$0 = \frac{\partial}{\partial q}\mathbb{E}\left[(q^{**}-q)^2\right] = \mathbb{E}\left[\frac{\partial}{\partial q}(q^{**}-q)^2\right] = 2\mathbb{E}[q^{**}-q],$$

so $q = \mathbb{E}[q^{**}] = q^*$, which is the quota we set with cap-and-trade with uncertainty.

Whenever we choose price $\tilde{p}$, the quantity is

$$q = MPV^{-1}(\tilde{p}) \Rightarrow q = q^* + \frac{p^*-\tilde{p}}{a} + \varepsilon.$$

We still want to minimize the expected loss, but by choosing price $\tilde{p}$, (the rigged part)

$$\begin{aligned}
\min_{\tilde{p}} \mathbb{E}\left[\frac{1}{2}(a+b)\left(q^{**} - \left(q^* + \frac{p^*-\tilde{p}}{a} + \varepsilon\right)\right)^2\right] \quad &\Leftrightarrow \quad \min_{\tilde{p}} \mathbb{E}\left[\left(\frac{p^*-\tilde{p}}{a}\right)^2 + 2\frac{b}{a+b}\varepsilon\frac{p^*-\tilde{p}}{a} + \frac{b^2}{(a+b)^2}\varepsilon^2\right] \\
&\Leftrightarrow \quad \min_{\tilde{p}} \mathbb{E}\left[\left(\frac{p^*-\tilde{p}}{a}\right)^2\right]
\end{aligned}$$

Therefore, $\tilde{p}^* = p^*$ and

$$q = \mathbb{E}\left[MPV^{-1}(\tilde{p}^*)\right] = \mathbb{E}[q^{**}+\varepsilon] = q^*.$$

So the optimal price we choose is $\tilde{p}^*$ is p^*. However, at p^*, the quantity is $q = q^* + \varepsilon$.

To recap, in the cap and trade, the effective quanity is $q^{CT} = q^*$ but with Pigouvian tax, the effective quantity is $q^{PT} = q^* + \varepsilon$. The better case is whichever has a smaller Harberger triangle. Whichever policy that results in a smaller Harberger triangle is the one with smalelr $|q^{**} - q|$. To compare,

$$\begin{aligned}
\left|q^{**} - q^{CT}\right| &= \frac{a}{a+b}|\varepsilon|, \\
\left|q^{**} - q^{PT}\right| &= \frac{b}{a+b}|\varepsilon|.
\end{aligned}$$

If $a < b$, cap-and-trade (quantity control) is better than price control.

8 Incidence

(Probably) To be added.

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