201 Notes: Product Design and Price Discrimination

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Thursday, November 15, 2012

I try to clarify the mathematics hiding amidst the lecture notes and use some examples from the current problem set to delineate the two topics covered in the lectures: multidimensional product design and price discrimination.

1 Product Design

Let's start from the basic model of Spence [1975]. Suppose there are many consumers. Each consumer gets utility $u(\rho,\theta)-P$. Consumer's type is distributed according to $f(\theta)$. Anyone with $u \geq P$ will buy and will not otherwise. Therefore, the total number of purchasers is,

$$N(P,\rho) = \int_{\theta:u(\rho,\theta)-P\geq 0} f(\theta)\, d\theta,$$

We define $\Theta(P,\rho) \equiv \{\theta : u(\rho,\theta)-P \geq 0\}$ to be the set of buyers who have positive utility and will make the purchase. The cost function of a firm is $C(N,\rho)$, and the firm's profit is revenue minus cost,

$$\pi(P,\rho) = P \cdot N(P,\rho) - C(N(P,\rho),\rho) \tag{1}$$

The social welfare

$$SW(P,\rho) = \int_{\{\theta:u(\rho,\theta)-P\geq 0\}} u(\rho,\theta) f(\theta)\, d\theta - C(N,\rho), \tag{2}$$

Let's understand the **Leibnitz Rule (LR)** first. It is stated as follows.

$$\frac{\partial}{\partial y}\int_{\{y:g(x,y)\geq 0\}} f(x,y)\, dx = \int_{\{y:g(x,y)=0\}} g_y(x,y) f(x,y)\, dx + \int_{\{y:g(x,y)\geq 0\}} f_y(x,y)\, dx.$$

How does this get specified to become the "Leibnitz Rule" I presented in last week's TA session and the **Fundamental Theorem of Calculus**?

$$\frac{\partial}{\partial y}\int_{a(y)}^{b(y)} f(x,y)\, dx = f(b(y),y)\, b'(y) - f(a(y),y)\, a'(y) + \int_{a(y)}^{b(y)} f_y(x,y)\, dx.$$

*All errors are solely mine. For questions, please email hanzhe@uchicago.edu.

To map to the most general formula above, we have

$$\begin{aligned}
& \frac{\partial}{\partial y}\left[\int_{a(y)}^{b(y)} f(x,y)\,dx\right] \\
= & \frac{\partial}{\partial y}\left[\int_{x\geq a(y)} f(x,y)\,dx-\int_{x\geq b(y)} f(x,y)\,dx\right] \\
= & \int_{x-a(y)\geq 0} f_y(x,y)\,dx+\int_{x=a(y)} -a'(y)\,f(x,y)\,dx-\int_{x-b(y)\geq 0} f_y(x,y)\,dx-\int_{x=b(y)} -b'(y)\,f(x,y)\,dx \\
= & \int_{x-a(y)\geq 0} f_y(x,y)\,dx-\int_{x-b(y)\geq 0} f_y(x,y)\,dx+b'(y)\,f(b(y),y)-a'(y)\,f(a(y),y)
\end{aligned}$$

And setting $b(y)=y$, $a(y)=0$, and $f(x,y)=f(x)$ yields the Fundamental Theorem of Calculus,

$$\frac{\partial}{\partial y}\int_0^y f(x)\,dx=f(y).$$

For what we do subsequently, you have to be comfortable with the most general form of the Leibnitz Rule.

Define $\partial\Theta(P,\rho)=\{\theta:u(\rho,\theta)=P\}$ to be the set of marginal consumers. , so the firm's profit is $PN-C(N,\rho)$. Fixing N and choosing ρ, the first order condition of the firm's problem is

$$\begin{aligned}
\frac{dP(\rho^m)}{d\rho}N+\frac{dN(P(\rho^m),\rho^m)}{d\rho}P-C_\rho(N(P(\rho^m),\rho^m),\rho^m) &= 0. \\
\frac{\partial P(\rho^m)}{\partial\rho}N-C_\rho(N(P(\rho^m),\rho^m),\rho^m) &= 0
\end{aligned}$$

Fixing N,

$$\frac{dN(P(\rho^m),\rho^m)}{d\rho}=\frac{\partial N(P(\rho^m),\rho^m)}{\partial P}\cdot\frac{\partial P(\rho^m)}{\partial\rho}+\frac{\partial N(P(\rho^m),\rho^m)}{\partial\rho}=0$$

and by LR,

$$\begin{aligned}
\frac{\partial N}{\partial P} &= \int_{\partial\Theta}(-1)\,f(\theta)\,d\theta \\
\frac{\partial N}{\partial\rho} &= \int_{\partial\Theta}u'(\rho,\theta)\,f(\theta)\,d\theta
\end{aligned}$$

so

$$\frac{\partial P(\rho^m)}{\partial\rho}=-\frac{\partial N/\partial P}{\partial N/\partial\rho}=\frac{\int_{\partial\Theta}u'(\rho,\theta)\,f(\theta)\,d\theta}{\int_{\partial\Theta}f(\theta)\,d\theta}=\int_{\partial\Theta}u'(\rho,\theta)\,\frac{f(\theta)}{\int_{\partial\Theta}f(\theta)\,d\theta}d\theta=\mathbb{E}\left[u'(\rho,\theta)\,|\theta\in\partial\Theta\right].$$

The monopoly quality ρ^m satisfies that

$$N\mathbb{E}\left[u'(\rho^m,\theta)\,|\theta\in\partial\Theta(\rho^m,\theta)\right]=C_\rho(N,\rho^m)$$

And for the social planner's problem, fixing N, the FOC (by LR) is

$$\begin{aligned}
\int_\Theta u'(\rho^*,\theta)\,f(\theta)\,d\theta+\int_{\partial\Theta}u'(\rho^*,\theta)\,u(\rho^*,\theta)\,f(\theta)\,d\theta-C_\rho(N,\rho^*) &= 0 \\
\frac{\int_\Theta u'(\rho^*,\theta)\,f(\theta)\,d\theta}{\int_\Theta f(\theta)\,d\theta}\cdot N+\int_{u(\rho^*,\theta)=P}u'(\rho^*,\theta)\,u(\rho^*,\theta)\,f(\theta)\,d\theta-C_\rho &= 0 \\
N\mathbb{E}\left[u'|\theta\in\Theta\right]-C_\rho &= 0.
\end{aligned}$$

Therefore, the socially optimal quality ρ^* satisfies

$$N\mathbb{E}\left[u'(\rho^*,\theta)|\theta\in\Theta(\rho^*,\theta)\right]=C_\rho(N,\rho^*).$$

Now if the marginal consumers get more marginal utility of quality than the average consumers,

$$N\mathbb{E}\left[u'(\rho^*,\theta)|\partial\Theta(\rho^*,\theta)\right]>N\mathbb{E}\left[u'(\rho^*,\theta)|\Theta(\rho^*,\theta)\right]=C_\rho(N,\rho^m),$$

then the monopoly's choice of quality $\rho^m>\rho^*$. This depends on the fact that u is concave in ρ and C is convex in ρ. The difference between the monopoly quality and social planner's choice of quality is the Spence distortion where quality provided is too high (low) if marginals value it more (less) than the infra-marginal. In this case, if the marginal consumer values advertising more than the infra-marginal (in the first case), then the advertising provided is too high, and it goes the opposite way when marginal type is the most educated among all. This point directly applies to Question 1.

Question (2012PS4LA2b). *Suppose that individuals differ only in their level of education $e>0$. Their utility from reading a newspaper is $(\alpha-\beta A)e-P$, where $\alpha,\beta>0$ and A is the level of advertising in the paper while P is its price. Assuming at least someone buys the newspaper, if price is positive, as it is for most newspapers, which part of the population reads the newspaper? Assuming not everyone reads it, if the newspaper is given out for free in an aggressive manner so it essentially has a negative price, who will read it? In each of these cases, will advertising tend to be excessive or insufficient from a social perspective, holding fixed the number of individuals who read?*

Solution. *When $P>0$, if someone reads the newspaper, then $0<P<(\alpha-\beta A)e$ for some $e>0$. In this case, the marginal type is less educated than the infra-marginal/average reader. Since the marginal disutility from advertising is βe, thus the more educated a person is, the more disutility he gets from advertising. Since the marginal type is the least educated and he is the least averse to advertising among the readers, so the monopoly newspaper is going to choose an advertising level that is despised by the higher type, thus providing excessive advertising.*

Therefore, we must have that $\alpha-\beta A>0$ and all types $e\geq P/(\alpha-\beta A)$ will read the newspaper. If $P<0$ and still not everyone reads it, then it must be that $\alpha-\beta A<0$ and $e\leq P/(\alpha-\beta A)>0$, the less educated will read the newspaper. Since the marginal consumer now is the most educated among all, advertising is insufficient.

Mussa-Rosen: Previously we have the cost function to be a function of ρ and N, however, it is possible that the cost depends on the utility: the more a person enjoys it, the more frequently she buys. Therefore, the cost function is $C(\rho;u(\rho,\theta))$. The right hand side of the equilibrium equation is now,

$$\begin{aligned}
&\frac{\partial\int_\Theta C(\rho,u(\rho,\theta))\,d\theta}{\partial\rho}\\
=\ &\int_\Theta\left[C_\rho(\rho,u(\rho,\theta))+C_u u'\right]f(\theta)\,d\theta+\int_{\partial\Theta}u'Cf(\theta)\,d\theta\\
=\ &N\mathbb{E}\left[C_\rho+u'C_u|\Theta\right]
\end{aligned}$$

if N is held fixed, the second term is zero. If $C_u<0$, can lower cost from intensive uses of the most loyal users.

Veiga and Weyl [2012]: Suppose that $u(\rho,\theta)=\theta$, holding N fixed,

$$\begin{aligned}
\frac{\partial\int_\Theta C(\rho,u(\rho,\theta))\,d\theta}{\partial\rho} &= M\mathbb{E}\left[u'C|\partial\Theta\right]-M\mathbb{E}\left[u'|\partial\Theta\right]\mathbb{E}\left[C|\partial\Theta\right]+N\mathbb{E}\left[C'|\Theta\right]\\
&= M\mathrm{cov}\left(u',C|\partial\Theta\right)+N\mathbb{E}\left[C'|\Theta\right].
\end{aligned}$$

where $M\mathrm{cov}(u', C|\partial\Theta)$ is the sorting cost of ρ, because it characterizes how at the margin, cost and marginal value of quality are correlated. If the marginal consumer with high value for quality is more costly to satisfy, then the sorting cost is high. Therefore, in addition to the intensive margin effect of Mussa-Rosen, Veiga and Weyl [2012] shows an extensive margin effect.

Question (2012PS4LA2a)**.** *Suppose that individuals differ along two dimensions, the cost of serving the C per unit of service provided q and the ratio of the value they get from the service to the cost of serving them, v. Individuals are therefore willing to pay $vC\rho$ for service of quality ρ and it costs $C\rho$ to provide this service. Suppose only one quality level can be offered. What is the sorting value of raising this quality level.*

Solution. *The sorting value is the covariance between the marginal utility for quality and their value to the firm, along the set of marginal consumers*

$$cov\left(\frac{\partial u}{\partial \rho}, C|u = P\right).$$

In this particular question, $u = vC\rho$, so $du/d\rho = vC$, but there is only one product ρ and one price P, so $vC = P/\rho$. Since it is a constant, the covariance is zero, so there is no value.

2 Price Discrimination

In the subsequent questions, we illustrate price discrimination is not necessarily bad, as social welfare, quantity produced unambiguously improve with price discrimination and consumer surplus is increased under considerable portion of circumstances.

Question (2012PS4LA2c)**.** *Suppose individuals have a willingness-to-pay for a good that is uniformly distributed on $[0, v]$. What price does a monopolist charge if her product is free to produce? What are consumer welfare, output profits, and social surplus?*

Solution. *Since it's uniformly distributed, the probability density function is $f(x) = 1/v$, and $F(x) = x/v$. The expected profit of setting the price at p is then*

$$(1 - F(p))\,p$$

FOC is

$$1 - F(p^*) - f(p^*)\,p^* = 0 \Rightarrow p^* - \frac{1 - F(p^*)}{f(p^*)} = 0 \Rightarrow p^* - \frac{1 - \frac{p^*}{v}}{\frac{1}{v}} = 0 \Rightarrow p^* = \frac{v}{2}.$$

Then the consumer surplus is

$$\int_p^v (x - p)\,f(x)\,dx = \frac{1}{v}\frac{x^2}{2}\bigg|_p^v - [F(v) - F(p)]\,p = \frac{1}{2}\frac{1}{v}\left(v^2 - \frac{v^2}{4}\right) - \left[1 - \frac{1}{2}\right]\frac{v}{2} = \frac{3}{8}v - \frac{1}{4}v = \frac{1}{8}v$$

Producer surplus is

$$\int_p^v p f(x)\,dx = p\,[F(v) - F(p)] = \left(1 - \frac{1}{2}\right)\frac{v}{2} = \frac{1}{4}v,$$

and social welfare is adding up two together, which is $\frac{3}{8}v$.

Question (2012PS4LA2d). *Now continuing part 2, suppose individuals can be divided into two groups, those with willingness-to-pay above $v^* \in (0,v)$ and those below. What is the optimal price in each market, consumer surplus, output, profits and social surplus? In which sense is discrimination beneficial or harmful? How does this depend, or not, on the value of v^*?*

Solution. *The profits of the two markets for choices of p_1 and p_2 are*

$$\begin{aligned}\pi_1(p_1) &= p_1(1-F(p_1))\\ \pi_2(p_2) &= p_2(F(v^*)-F(p_2))\end{aligned}$$

So the marginal profit is

$$\begin{aligned}\pi_1'(p_1) &= 1-F(p_1)-p_1 f(p_1) = 1-2p_1\\ \pi_2'(p_2) &= F(v^*)-F(p_2)-p_2 f(p_2) = v^*-2p_2\end{aligned}$$

If $v^* > v/2$, then $p_1^* = v^*$; otherwise, $p_1^* = v/2$. Regardless of v^*, $p_2^* = v^*/2$. The producer surplus is

$$\begin{aligned}\pi_1(p_1^*)+\pi_2(p_2^*) &= p_1^*\left(1-\frac{p_1^*}{v}\right)+p_2^*\left(v^*-\frac{p_2^*}{v}\right)\\ &= \begin{cases} v^*\left(1-\frac{v^*}{v}\right)+\frac{v^*}{2}\left(\frac{v^*}{v}-\frac{1}{2}\frac{v^*}{v}\right)=v^*-\frac{3}{4}\frac{v^{*2}}{v} & : v^* > v/2\\ \frac{v}{2}\left(1-\frac{1}{2}\right)+\frac{v^*}{2}\left(1-\frac{v^*}{2v}\right)=\frac{1}{4}+\frac{1}{2}\left(v^*-\frac{v^{*2}}{2v}\right) & : v^* < v/2\end{cases}\end{aligned}$$

The social welfare is just

$$\begin{aligned} & \int_{v^*}^{v} x\cdot\mathbf{1}_{x>p_1^*} f(x)\,dx+\int_0^{v^*} x\cdot\mathbf{1}_{x>p_2^*} f(x)\,dx\\ v^* > \frac{v}{2}: &= \frac{1}{v}\left(\int_{v^*}^{v} x dx+\int_{v^*/2}^{v^*} x dx\right)=\frac{1}{v}\frac{x^2}{2}\bigg|_{\frac{v^*}{2}}^{v}=\frac{1}{2v}\left(v^2-\frac{v^{*2}}{4}\right)=\frac{1}{2}v-\frac{1}{8}\frac{v^{*2}}{v}\geq\frac{1}{8}v\\ v^* < \frac{v}{2}: &= \frac{1}{v}\left(\int_{v/2}^{v} x dx+\int_{v^*/2}^{v^*} x dx\right)=\frac{3}{8}\frac{1}{v}\left(v^2+{v^*}^2\right)=\frac{3}{8}v+\frac{3}{8}v^{*2}/v\end{aligned}$$

The consumer surplus is ambiguous.

This is related to the TFU4.

References

Michael A. Spence. Monopoly, quality, and regulation. *The Bell Journal of Economics*, pages 417–429, 1975.

Andre Veiga and Glen Weyl. Multidimensional product design. Working Paper, 2012.