ECON20710 Lecture
Auction as a Bayesian Game

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Introduction

Auction theory has been a particularly successful application of game theory ideas to the real world, with its use ranging from eBay auctions on personal items such as phones and laptops worth few hundred dollars, to those which sell treasury bonds, rights to use radio spectrums, and schedule and allocate courses.

We focus on the simplest auction: first price auction (FPA) to sell one unit of good with the simplest assumptions on buyers' values: independent, private values. First, we show that FPA can be modeled as a Bayesian game. We get a glimpse of equilibrium bidding strategies look like from an example in which two players have finite number of possible values and can only bid one of the finitely many prices. Next, we show how we derive a symmetric equilibrium when the buyer's type set is continuous (thus infinite). Related auction formats like second price and Dutch auctions are mentioned, and the celebrated Revenue Equivalence Theorem and its relation with Revelation Principle are briefly touched upon. References for further readings and exercises are left for interested and dedicated readers.

1 Bayesian Game

What is a Bayesian game?

Definition 1. A **Bayesian game** $(N,\Omega,A,S,\tau,\Pr,u)$ consists of

- a set of n **players**, $N=\{i\}$,
- a set of **states**, $\Omega \ni \omega$,

and for each player i,

- a set of **actions** $A_i \ni a_i$,
- a set of **signals** $S_i \ni s_i$, that she may receive and a **signal function** $\tau_i : \Omega \to S_i$ that associates a signal with each state,
- for each signal that she may receive, a **belief** about the states consistent with the signal (a probability distribution over the set of states with which the signal is associated), and
- a **payoff function** $u_i : A \times \Omega \to \mathbb{R}$ representing the preferences/utilities over the action-state pairs, and expected utility $u_i : A_i \to \mathbb{R}$.

*All errors are solely mine. For questions and suggestions, please email hanzhe@uchicago.edu.

What is a Nash equilibrium of a Bayesian Game?

Definition 2. A **Nash equilibrium of a Bayesian Game/Bayesian-Nash equilibrium** is a Nash equilibrium of the corresponding strategic game (with vNM preferences) with

- set of players as the set of all pairs (i, s_i) where i is the player in the Bayesian game and s_i one of the signals i may receive,
- the set of actions of each player (i, s_i) is the set of actions of player i in the Bayesian game, and
- the Bernoulli payoff function of each player (i, s_i) is

$$u_{(i,s_i)}(a_i) = \sum_{\omega \in \Omega} \Pr(\omega|s_i)\, u_i\left((a_i, a_{-i}(\omega)), \omega\right).$$

Essentially, the Nash equilibrium in the Bayesian game is a strategy profile $\left\{a^*_{(i,s_i)}\right\}$ such that for each (i, s_i), $a^*_{(i,s_i)}$ is the best response to $a^*_{-(i,s_i)}$ that maximizes the expected utility

$$u_{(i,s_i)}\left(a^*_{(i,s_i)}\right) = \sum_{\omega \in \Omega} \Pr(\omega|s_i)\, u_i\left(\left(a^*_i, a^*_{-i}(\omega)\right), \omega\right).$$

. Next, we see how a first price auction is a Bayesian game and solve the symmetric Nash equilibrium in it.

2 First Price Auction with Discrete Types

In this section, we look at a setting where players have finite number of possible values and finite number of possible strategies. We "guess" an equilibrium strategy first and verify that the strategy is indeed an equilibrium because fixing other players' strategy, a player's strategy is the best response that maximizes the expected utility over the belief distribution.

Two players bid for a painting. Each player has a private value v_i to the painting only known by herself and everyone else only knows that it is either 2 or 3 with equal probability. The rule of the **first price auction** is that the player with the highest bid wins and pays her bid with ties broken by flip of a coin. Suppose that the players can only bid integers 0, 1, 2, or 3.[1]

How does this game map to a Bayesian game?

- There are two $(n = 2)$ players, $N = \{1, 2\}$.
- The state set Ω is denoted by the values of players, $\omega = (v_1, v_2) \in \{(2,2), (2,3), (3,2), (3,3)\}$.
- For each player, the set of actions is $A_i = \{0, 1, 2, 3\}$.
- The signal s_i player i receives is 2 or 3, and her value of the good is exactly the signal she receives, $v_i = s_i$ (*private* value).
- For any signal she receives, her belief about the state is basically the belief of her opponent's signal, and it is the same conditional on the same signal she receive (*independent*)., i.e.

$$\Pr(\omega|v_i) = \Pr(v_j|v_i) = \Pr(v_j) = \frac{1}{2} \forall v_j \in \{2, 3\}$$

[1] This example is borrowed from Jackson and Shoham's Coursera lecture on Discrete First Price Auctions.

- The utility function given the action and state, with the rule of FPA,

$$u_i\left(\left(a_i,a_j\right),\left(v_i,v_j\right)\right)=\begin{cases}0 & : a_i<a_j\\ \frac{1}{2}\left(v_i-a_i\right) & : a_i=a_j\\ v_i-a_i & : a_i>a_j\end{cases}$$

Now let us solve for a Nash equilibrium in this game. The corresponding strategic form game is demonstrated as follows. Since the game is symmetric, we show one player's payoffs and actions.

$s_i = 2$

$a_i \backslash a_j$	0	1	2	3
0	1	0	0	0
1	1	$\frac{1}{2}$	0	0
2	0	0	0	0
3	-1	-1	-1	$-\frac{1}{2}$

$s_i = 3$

$a_i \backslash a_j$	0	1	2	3
0	$\frac{3}{2}$	0	0	0
1	2	1	0	0
2	1	1	$\frac{1}{2}$	0
3	0	0	0	0

Table 1: The corresponding strategic game

Guess

Let us look for a symmetric pure strategy Nash equilibrium in this Bayesian game, that is, we find a pair $b^*=(b(2),b(3))$ where player $(i,2)$ will bid $b(2)$ and player $(i,3)$ will bid $b(3)$ for any i.

If a player bids her value, i.e., $b(2)=2$, $b(3)=3$, the payoff is 0 no matter she wins or loses. This is the same payoff as not participating, the worst payoff for a "rational" player who never bids higher than her own value, as any other bid lower than value results in possible positive payoff without loss. Maybe one can do better by **shading** - bidding lower than her own value.

Verify

Try $b(3)=2$, $b(2)=1$. We show that this is indeed a symmetric Nash equilibrium. Succinctly, this bidding function is $b^*(v)=v-1$. Given that the other player will bid according to $b^*(\cdot)$, a player with value 3 will get the respective payoffs when she chooses different actions a_i:

$$\begin{aligned} u_i(0) &= u_i(3)=0\\ u_i(1) &= \frac{1}{2}\cdot 1+\frac{1}{2}\cdot 0=\frac{1}{2}\\ u_i(2) &= \frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}\end{aligned}$$

On the other hand, if the player has value 2, the payoffs when she bids a_i is

$$\begin{aligned} u_i(0) &= u_i(2)=0\\ u_i(1) &= \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot 0=\frac{1}{4}\\ u_i(3) &< 0.\end{aligned}$$

Therefore, indeed $b^*(\cdot)$ is a symmetric equilibrium.

Summary

There are two points to be taken from this example. First, FPA with discrete types can be a Bayesian game. Second, in equilibrium a bidder bids lower than her true value to increase her expected payoff. Although it's possible that the final transaction is below a buyer's willingness to pay (thus triggering some sort of ex-post regret of not bidding), it is nonetheless best response to bid lower than own value to maximize ex-ante expected payoff. We show next that both points stay true when the signal and type sets are continuous.

Furthermore, let's recap how we've gone this far. First, we formulate the auction problem as a Bayesian game. Then, we see that bidding exactly one's value seems to be suboptimal, so we guess an equilibrium that both types of bidders shade their bids. Nonetheless, the equilibrium bidding function is strictly increasing (type 3 bids 2, type 2 bids 1) and symmetric. Finally, we verify that it is indeed an equilibrium.

3 First Price Auction with Continuous Type Set[2]

Suppose now that there are n buyers each observe a signal which is drawn from the distribution with cumulative distribution function F on $[0,1]$ with strictly increasing density function f. This latter assumption guarantees that the probability of a tie is zero. A bidder's value is still her signal. It is still a Bayesian game with

- n players,
- state set with each element denoted by the values of the bidders, $\omega = (v_1, \cdots, v_n)$,
- player i's action set $b_i \in \mathbb{R}$,
- a private signal s_i player i, and $s_i = \tau_i(\omega)$,
- player i's belief about the state $\Pr_i(\omega | v_i) = \times_{j \neq i} f(v_j)$,
- the utility of player i of value v_i submitting b_i is

$$u_i((b_i, b_{-i}), (v_i, v_{-i})) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j \text{ and there are } k-1 \text{ ties} \\ 0 & \text{otherwise.} \end{cases}$$

Now we try find a Nash equilibrium that mimics the guess-and-verify approach used in the previous example. First we try to guess that there exists a *symmetric, strictly increasing bidding function* as the Nash equilibrium. By definition, given that every other bidder j bids according to bidding function $b_j(\cdot)$, bidder i of value v_i should bid $b_i(v_i)$. Since it's symmetric, we can drop the subscript i.

Guess

Let the equilibrium bidding function of agent with value v be $b(v)$. The expected value of bidder of value v bidding a when everyone else bids according to $b(\cdot)$,

$$u(a|v) = \Pr(a > b(v_j) \forall j \neq i) \cdot (v - a) = F^{n-1}\left(b^{-1}(a)\right)(v-a). \tag{1}$$

[2] The first derivation follows Jehle and Reny (2011) (JR2 hereafter) and the second Osborne (2004).

Calculate: Approach 1

Now a thought experiment. Any other choice of a is pretending to be bid as if you are some other type r, i.e., $a = b(r)$. Therefore, choosing a bid a is equivalent to choosing the type r that a bidder mimics, as illustrated in Figure 1. Furthermore, there is no incentive to have $a > b(1)$, because bidding higher than $b(1)$ will not result in higher probability of winning but only causes higher payment than $b(1)$.

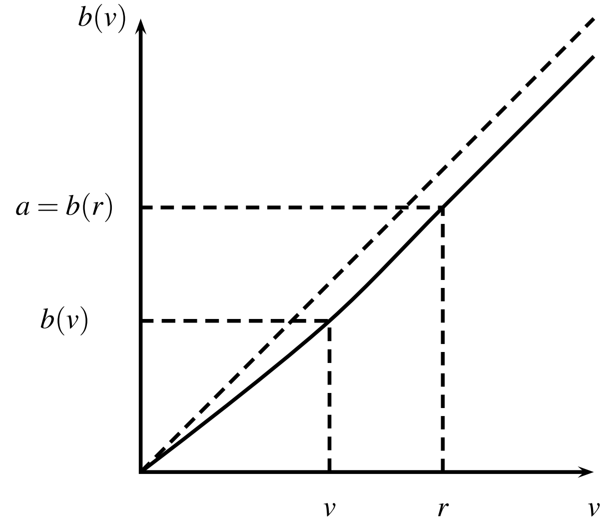

Figure 1: Choosing a is equivalent to choosing $b(r)$, which is equivalent to choosing r

Then the expected utility function can be written as

$$u(b(r), v) = F^{n-1}(r)(v - b(r)).$$

By definition of a Nash equilibrium, the best response of bidder with value v to others' equilibrium bids is to bid $b(v)$, that is,

$$u(b(v), v) \geq u(b(r), v) \forall r$$

Maximizing the expected utility with respect to r,

$$\frac{\partial u(b(r), v)}{\partial r} = \left(F^{n-1}(r)\right)'(v - b(r)) - F^{n-1}(r) b'(r) \tag{2}$$

and FOC is satisfied at $r = v$, that is,

$$\begin{aligned} 0 &= \left(F^{n-1}(v)\right)'(v - b(v)) - F^{n-1}(v) b'(v) \quad (3)\\ 0 &= \left(F^{n-1}(v)\right)' b(v) - F^{n-1}(v) b'(v) + (n-1) F^{n-2}(r) f(v) v \\ 0 &= (n-1) F^{n-2}(v) f(v) v - d\left(F^{n-1}(v) b(v)\right)/dv \end{aligned}$$

In differential form and solve,

$$\begin{aligned} d\left(F^{n-1}(v) b(v)\right) &= (n-1) F^{n-2}(v) f(v) v dv \\ F^{n-1}(v) b(v) &= \int_0^v (n-1) \cdot F^{n-2}(x) f(x) x dx + K \end{aligned}$$

And we have another equilibrium boundary condition $b(0) = 0$, that is, equilibrium bid by value 0 player is 0. Therefore for $v = 0$, we can plug in $b(v)$ and get that $K = 0$. Written more succinctly, the **equilibrium bidding function** is

$$b(v) = \frac{1}{F^{n-1}(v)} \int_0^v x dF^{n-1}(x) = \mathbb{E}\left[\max_{j \neq i} v_j \,\middle|\, \max_{j \neq i} v_j < v\right] \tag{4}$$

Calculate: Approach 2

Instead of going through the thought experiment, we can directly make the choice of a to maximize the expected utility from equation 1 by taking the FOC. It is more convoluted mathematically than the first approach but may be more straightforward economically.

$$\begin{aligned} \frac{\partial\left[F^{n-1}\left(b^{-1}(a)\right)(v-a)\right]}{\partial a} &= -F^{n-1}\left(b^{-1}(a)\right) + (v-a)(n-1)F^{n-2}\left(b^{-1}(a)\right) f\left(b^{-1}(a)\right) \frac{\partial b^{-1}(a)}{\partial a} \\ &= -F^{n-1}\left(b^{-1}(a)\right) + (v-a)(n-1)F^{n-2}\left(b^{-1}(a)\right) f\left(b^{-1}(a)\right) \frac{1}{b'\left(b^{-1}(a)\right)}. \end{aligned}$$

At the utility-maximizing choice a^*, $b^{-1}(a^*) = v$, that is, the person who bids a^* has value v, and $a^* = b(v)$ and FOC is 0.

$$-F^{n-1}(v) + (v - b(v))(n-1)F^{n-2}(v)\frac{1}{b'(v)} = 0$$

which is the same as we have arrived in the previous approach.

Example 1. If $v_i \sim U[0,1]$, i.e. $F(v_i) = v_i, f(v_i) = 1 \forall i$, then the equilibrium strategy in Equation 4 is

$$b(v) = \frac{1}{v^{n-1}} \int_0^v x(n-1)x^{n-2} dx = \frac{1}{v^{n-1}}\left[\frac{n-1}{n} x^n \Big|_0^v\right] = \frac{n-1}{n} v.$$

The economic interpretation of this bidding function is that the bidder will *bid the expectation of the highest value among opponents assuming that a bidder wins* (i.e., has the highest bid, or has the highest value). The idea that one ought to bid conditional on winning is a key feature and insight in strategic analysis of bidder behaviors. In the exercise, you are asked in exercise to show that the equilibrium bidding exhibits shading

$$b(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{n-1} dx.$$

Verify

So far we only solve out a candidate equilibrium bidding function. We have just guessed that the bidding function is symmetric and increasing and the candidate equilibrium bidding function is in the form illustrated above. However, we have not shown that it is indeed an equilibrium. Exercise 2 guides you to verify this is indeed the case.

Expected Revenue

The expected revenue of the seller from running this FPA is the expected bid of the highest value bidder,

$$\int_0^1 \left[\frac{1}{F^{n-1}(v)} \int_0^v x dF^{n-1}(x)\right] dF^n(v) = \mathbb{E}\left[\mathbb{E}\left[\max_{j \neq i} v_j \,\middle|\, \max_{j \neq i} v_j < v\right] | v = \max v_j\right].$$

4 Second Price Auctions, Open-Cry Auctions, and Revenue Equivalence Theorem

First price auction is a sealed-bid auction where each player essentially puts a bid in a sealed envelope and submit it to the seller. There is a set of auctions where they are conducted in the auction house, where people actively participate while visible to each other. In particular, **Dutch auction**, is an auction in which price starts from Dutch auction is actually equivalent to the first price sealed-bid auction.

Second price auction is a sealed-bid auction where the player with the highest bid pays *the highest bid among losers*. It turns out that bidding one's own value is the dominant strategy. However, perhaps more surprisingly is that the equal expected revenue generated by the FPA and SPA! In fact, (first-price and second-price) all-pay auctions will generate the same expected revenues too! This is the celebrated **Revenue Equivalence Theorem**. The proof depends on characterizing all the auctions as a pair of functions from reports: probability assignment function and payment function, and utilizing the **Revelation Principle** that we previously mentioned in the straightforward games.

There are some classic papers that are must-read. **?** shows that the second price auction has truthful report as the dominant strategy. Myerson (1981) shows that first- and second-price auctions (among many others) have the same expected revenue, and more importantly, the auction that restricts participation to buyers with high values is generated the most expected revenue among ALL selling mechanisms (posting prices, lotteries, whatever you can think of). Furthermore, it generalizes Vickrey's observation that expected revenue of FPA and SPA is equal. Milgrom and Weber (1982) extends the analysis to settings where the buyers' values are affiliated; for example, oil field in which no one actually knows the exact amount of oil but comes up with separate estimates that are correlated.

For textbook treatment, pages 80-91, 291-301, and 307-310 of Osborne (2004) have a nice introductory guide to the basic first-price and second-price auctions as applications of a Bayesian game (as we have covered). For a little more in-depth analysis on Revenue Equivalence Theorem and reserve price, Jehle and Reny (2011) is a good guide. Krishna (2002) is a book entirely devoted to auction, suitable for advanced undergraduates. Milgrom (2004) is a more compact and advanced text that connects with research frontiers.

5 Exercises

5.1 Basic Exercises

Exercise 1 (JR2 9.1)**.** Show that the equilibrium bidding function

$$b(v) = \frac{1}{F^{n-1}(v)} \int_0^v x dF^{n-1}(x)$$

is strictly increasing.

Proof. There are three similar ways of showing that the bidding function is strictly increasing, only differing in algebraic complexities. $\square$

1. Take the derivative with respect to v,

$$\begin{aligned} b^{'}(v) &= -(n-1) F^{-n}(v) f(v) \int_0^v x dF^{n-1}(x) + F^{-n+1}(n-1) v F^{n-2}(v) f(v) \\ &= (n-1)\frac{f(v)}{F(v)}\left[v - F^{-n+1}(v)\int_0^v x dF^{n-1}(x)\right] = (n-1)\frac{f(v)}{F(v)}(v - b(v)) \end{aligned}$$

   Economically $b(v) < v$ for all $v > 0$ because all strategies $b(v) > v$ are strictly dominated by $b(v) = v$ because if there is positive probability of winning, the expected payment is bigger than v so the expected payoff is negative if a bidder bids higher than his value. Mathematically or directly from the interpretation of the conditional expected value on winning,

$$F^{-n+1}(v) \int_0^v x dF^{v-1}(x) = \int_0^v x d\left(\frac{F^{n-1}(x)}{F^{n-1}(v)}\right) < \int_0^v v d\left(\frac{F^{n-1}(x)}{F^{n-1}(v)}\right) = v.$$

2. $u(r, v) = F^{n-1}(r)(v - b(r))$ achieves a critical point r when $r = v$. Consequently,

$$b'(v) = \frac{1}{F^{n-1}(v)}(n-1) f(v) F^{n-2}(v - b(v)) = \frac{1}{F(v)}(n-1) f(v)(v - b(v)).$$

3. Let $G(v) = F^{n-1}(v)$ and $g(v) = G'(v)$, then

$$\begin{aligned} b(v) &= \frac{1}{G(v)} \int_0^v x dG(x) = \frac{1}{G(v)} \int_0^v x g(x) dx \\ b'(v) &= -\frac{g(v)}{G^2(v)} \int_0^v x dG(x) + \frac{1}{G(v)} v g(v) \\ &= \frac{g(v)}{G(v)}\left[v - \frac{1}{G(v)} \int_0^v x dG(x)\right] = \frac{g(v)}{G(v)}(v - b(v)). \end{aligned}$$

Exercise 2 (JR2 9.2)**.** Show in two ways that the symmetric equilibrium bidding strategy of a first price auction with n symmetric bidders each with values distributed according to F, can be written as

$$b(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{n-1} dx.$$

(Hint: For the first way, use solution from text and apply integration by parts. For the second way, use the fact that $F^{n-1}(r)(v-b(r))$ is maximized in r when $r=v$ and apply the envelope theorem to conclude that $d\left(F^{n-1}(v)(v-b(v))\right)/dv = F^{n-1}(v)$; now integrate both sides from 0 to v.)[3]

Proof. The two ways are as follows.

1. Integrate the bidding function by parts, with $u=x$, $du=dx$, and $dv=dF^{n-1}(x)$, $v=F^{n-1}(x)$ in $\int udv = uv - \int vdu$,

$$\begin{aligned} b(v) &= F^{1-n}(v)\left[\left[xF^{n-1}(x)\right]\Big|_0^v - \int_0^v F^{n-1}(x)\,dx\right] \\ &= F^{1-n}(v)\left[vF^{n-1}(v) - \int_0^v F^{n-1}(x)\,dx\right] = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{n-1} dx \end{aligned}$$

2. Since utility is maximized at $r=v$, we have the equality,

$$u(b(v),v) = F^{n-1}(v)(v-b(v)) = \max_{r\in[0,1]} F^{n-1}(r)(v-b(r))$$

By envelope theorem, with respect to v, the equation becomes

$$\begin{aligned} d\left(F^{n-1}(v)(v-b(v))\right)/dv &= F^{n-1}(v) \\ d\left(F^{n-1}(x)(x-b(x))\right) &= F^{n-1}(x)\,dx \\ \int_0^v d\left(F^{n-1}(x)(x-b(x))\right) &= \int_0^v F^{n-1}(x)\,dx \\ v-b(v) &= \frac{1}{F^{n-1}(v)}\int_0^v F^{n-1}(x)\,dx \end{aligned}$$

Rearrange to get the desired result.

[3] This form of the equilibrium bidding function highlights the amount by which bidders **shade** their bids (i.e. bid below their value). Note that it is optimal to engage in more shading if there are fewer bidders and less shading as the number of bidders increases. As $n\to\infty$,

$$b(v) = v - \int_0^{v-\varepsilon/2}\left(\frac{F(x)}{F(v)}\right)^{n-1} dx - \int_{v-\varepsilon/2}^{v}\left(\frac{F(x)}{F(v)}\right)^{n-1} dx$$

We have,

$$\begin{aligned} \left|\int_0^{v-\varepsilon/2}\left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| &\le \int_0^{v-\varepsilon/2}\left|\left(\frac{F(x)}{F(v)}\right)^{n-1}\right| dx \le \int_0^{v-\varepsilon/2}\left|\left(\frac{F\left(v-\frac{\varepsilon}{2}\right)}{F(v)}\right)^{n-1}\right| dx \\ \left|\int_{v-\varepsilon/2}^{v}\left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| &\le \int_{v-\varepsilon/2}^{v}\left|\left(\frac{F(x)}{F(v)}\right)^{n-1}\right| dx \le \int_0^{v-\varepsilon/2} 1dx = \varepsilon/2 \end{aligned}$$

Since $\frac{F\left(v-\frac{\varepsilon}{2}\right)}{F(v)} < 1$, $\left(\frac{F\left(v-\frac{\varepsilon}{2}\right)}{F(v)}\right)^{n-1} \to 0$ as $n\to\infty$ implies that

$$\exists n_0 : \quad \left|\left(\frac{F\left(v-\frac{\varepsilon}{2}\right)}{F(v)}\right)^{n-1}\right| \le \varepsilon/2 \;\forall n \ge n_0$$

Then, for all $n \ge n_0$,

$$|v-b(v)| \le \left|\int_0^{v-\varepsilon/2}\left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| + \left|\int_{v-\varepsilon/2}^{v}\left(\frac{F(x)}{F(v)}\right)^{n-1} dx\right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Exercise 3 (JR2 9.3)**.** This exercise will guide you through the proof that the equilibrium bidding function is in fact a symmetric equilibrium of the first-price auction. (Hint: First show that

$$du(r,v)/dr = (n-1)F^{n-2}(r)f(r)(v-r)$$

and conclude that $du(r,v)/dr$ is positive when $r<v$ and negative when $r>v$ so that $u(r,v)$ is maximized when $r=v$.)

Solution. Because it holds for all $du(v,v)/dr=0$, it must hold for $v=r$, that is,

$$0 = du(r,r)/dr = (n-1)F^{n-2}(r)f(r)(r-b(r)) - F^{n-1}(r)\cdot b'(r)$$

Then by eq. 2,

$$\begin{aligned}\frac{du(r,v)}{dr} &= \frac{du(r,v)}{dr} - \frac{du(r,r)}{dr}\\ &= \left[(n-1)F^{n-2}(r)f(r)(v-b(r)) - F^{n-1}(r)\cdot b'(r)\right]\\ &\quad - \left[(n-1)F^{n-2}(r)f(r)(r-b(r)) - F^{n-1}(r)\cdot b'(r)\right]\\ &= (n-1)F^{n-2}(r)f(r)(v-r)\end{aligned}$$

and since $F,f>0$, $du(r,v)/dr>0$ when $v>r$ and <0 when $v<r$, therefore, $u(r,v)$ attains maximum at $r=v$.

5.2 Extended Exercises

Exercise 4. Is there another Nash equilibrium in the discrete auction game in Section 2?

Solution. $b(1)=b(2)=1$ is another NE. From Table 1, we can see that playing $\{0,1,2,3\}$ against the strategy will yield payoffs $\{0,1/2,0,-1\}$ for type $s=2$ and payoffs $\{0,1,1,0\}$ for $s=3$. Therefore, playing 1 is the best response.

Exercise 5 (Auction with buyers who have sequential outside options, Zhang (2012))**.** One item is for sale using first price auction. There are n buyers who have private values v independently drawn from the same cumulative distribution $F(\cdot)$ (probability density function is denoted by $f(\cdot)$). Suppose that in the subsequent (second) period, the same item will be available to all buyers at price p with probability $\lambda(p)$, and buyer i will purchase the good if the price is lower than v_i, and gets utility v_i-p. Buyers do not discount.

1. What is the expected utility of a bidder of value v in the second period? What is her expected utility when she bids $b(r)$ and others are bidding according to strictly increasing, symmetric $b(\cdot)$ as well?

   Solution. The expected utility of a bidder in the second period is

   $$u_2(v) = \int_0^v (v-p)\lambda(p)\,dp.$$

   The expected utility when she bids $b(r)$ while others bid $b(\cdot)$ is

   $$\begin{aligned}u(b(r),v) &= F^{n-1}(r)(v-b(r)) + \left(1-F^{n-1}(r)\right)u_2(v)\\ &= F^{n-1}(r)\left[(v-u_2(v))-b(r)\right]+u_2(v)\end{aligned}$$

   If we define $\tilde{v}(v)=v-u_2(v)$, then the expected utility is

   $$u(b(r),v) = F^{n-1}(r)(\tilde{v}(v)-b(r))+u_2(v)$$

2. Derive the symmetric, strictly increasing equilibrium bidding functions of the bidders in the first price auction.

Solution. Given that it's been put in the form above, it's not hard to see that the equilibrium bidding function is very similar as before, by substituting $\tilde{v}(v)$ for v (or $\tilde{v}(x)$ for x), or explicitly,

$$\begin{aligned}\left(F^{n-1}(v)\right)'(\tilde{v}(v)-b(v))-F^{n-1}(v)b'(v) &= 0\\ \left(F^{n-1}(v)\right)'\tilde{v}(v)-\left(F^{n-1}(v)b(v)\right)' &= 0\end{aligned}$$

Take integral,

$$F^{n-1}(v)b(v)=\int_0^v \tilde{v}(x)\left(F^{n-1}(x)\right)'dx+K$$

and we know that $b(0)=0$. Then denote the equilibrium bidding function by $\tilde{b}(\cdot)$, we get

$$\tilde{b}(v)=\frac{1}{F^{n-1}(v)}\int_0^v \tilde{v}(x)\,dF^{n-1}(x).$$

3. How does the equilibrium bidding function change compared to the one where there is no sequential outside option (as derived in class)?

Solution. Since $\tilde{v}(v)=v-u_2(v)\leq v$, $\tilde{b}(v)\leq b(v)$.

4. Two Distributions:

   (a) Calculate the equilibrium bidding function when buyer's value distribution is uniformly distributed on $[0,1]$ and there is probability $1/2$ that the item is for sale at price $1/2$ in the next period. (Mathematically, $f(v)=1\forall v\in[0,1]$ and $\lambda(1/2)=1/2;=0$ otherwise). Compare it to the one derived in example.

   Solution. In this case, for $v\geq 1/2$,

$$\tilde{v}(v)=v-\frac{1}{2}\left(v-\frac{1}{2}\right)=\frac{1}{2}\left(v+\frac{1}{2}\right).$$

   Therefore,

$$\tilde{v}(v)=\begin{cases}\frac{1}{2}\left(v+\frac{1}{2}\right) & v\geq\frac{1}{2}\\ v & v<\frac{1}{2}\end{cases}$$

   And the equilibrium bidding function is for $v\geq\frac{1}{2}$,

$$\tilde{b}(v)\;=\;\frac{1}{v^{n-1}}\left(\int_0^{\frac{1}{2}} x\,dx^{n-1}+\int_{\frac{1}{2}}^{v}\frac{1}{2}\left(x+\frac{1}{2}\right)dx^{n-1}\right)$$

$$\tilde{b}(v)=\begin{cases}\frac{n-1}{n}v & v<\frac{1}{2}\\ \frac{1}{4}+\frac{1}{2}\frac{n-1}{n}v-\frac{1}{n}\frac{1}{v^{n-1}}\left(\frac{1}{2}\right)^{n+1} & v\geq\frac{1}{2}\end{cases}$$

(b) Calculate the equilibrium bidding function when buyer's value distribution is uniformly distributed on $[0,1]$ and there is uniform offer in the next period. (Mathematically, $f(v) = 1 = \lambda(v) \forall v \in [0,1]$). Compare it to the one derived in example.

Solution. Plug in,

$$u_2(v) = \int_0^v (v-p)\,dp = \left(vp - \frac{p^2}{2}\right)_0^v = \frac{v^2}{2},$$

so $\tilde{v}(v) = v - v^2/2$.

$$\begin{aligned}\tilde{b}(v) &= \frac{1}{v^{n-1}} \int_0^v \left(x - \frac{x^2}{2}\right)(n-1)x^{n-2}dx = \frac{n-1}{v^{n-1}} \int_0^v \left(x^{n-1} - \frac{x^n}{2}\right)dx \\ &= \frac{n-1}{v^{n-1}} \left(\frac{v^n}{n} - \frac{1}{2}\frac{v^{n+1}}{n+1}\right) = \frac{n-1}{n}v - \frac{v^2}{2}\frac{1}{n+1} = b(v) - \frac{1}{n+1}\frac{v^2}{2}\end{aligned}$$

5.3 Challenging Exercises

Exercise 6 (Auction with uncertain number of bidders, Harstad et al. (1990)). Suppose now that each bidder does not know the exact number of opponents she's facing, but g_n is the probability that there are n bidders, with the number of possible bidders ranging from 2 to N. Bidders have independent private values drawn from common distribution F as before.

1. What is the expected utility of value v bidder bidding $b(r)$ and others bid according $b(\cdot)$?

   Solution. The expected utility is

   $$u(b(r), v) = \sum_n g_n F^{n-1}(r)(v - b(r))$$

2. Derive the symmetric, strictly increasing equilibrium bid function.

   Solution. Take the derivative with respect to r,

   $$\begin{aligned}\frac{\partial u(b(r), v)}{\partial r} &= \sum_n g_n \left[\left(F^{n-1}(r)\right)' v - \left(F^{n-1}(r)\,b(r)\right)'\right] \\ \left.\frac{\partial u(b(r), v)}{\partial r}\right|_{r=v} &= \sum_n g_n \left(F^{n-1}(v)\right)' v - \sum_n g_n \left(F^{n-1}(v)\,b(v)\right)' = 0\end{aligned}$$

   Take the integral,

   $$\sum_n g_n \int_0^v x\,d\left(F^{n-1}(x)\right) - \left[\sum_n g_n F^{n-1}(v)\right] b(v) = 0$$

   Therefore, the equilibrium bidding function is

   $$b(v) = \sum_n g_n \int_0^v x\,d\left(F^{n-1}(x)\right) / \left[\sum_n g_n F^{n-1}(v)\right]$$

3. How does the equilibrium bid function relate to the one when there are n bidders for $n \in \{2, \cdots, N\}$? Express it in terms of the standard bid function with n players if possible.

Solution. The equilibrium bidding function can be rewritten as

$$\begin{aligned} b(v) &= \frac{1}{\sum_n g_n F^{n-1}(v)} \sum_n g_n F^{n-1}(v) \left[\frac{1}{F^{n-1}(v)} \int_0^v x d\left(F^{n-1}(x)\right) \right] \\ &= \sum_n \frac{g_n F^{n-1}(v)}{\sum_n g_n F^{n-1}(v)} b_n(v) \end{aligned}$$

where $b_n(v)$ is the equilibrium bid of v when there are n bidders. The equilibrium bid with uncertain number of bidders is a weighted average of equilibrium bids with certain number of bidders, with the weights as the probability of having n bidders in the auction given that v is the winner.

5.4 Bargaining

Exercise 7 (Question 473.1)**.**

Solution. There are two distinct subgames

1. Subgame starting with a proposal from 1: If 1 offers 2 more than $1 - x_1$, then 2 accepts making 1 worse off then when she gets x_1 by following her strategy. If 1 offers 2 more than $1 - x_1$, 2 rejects, so 1 gets x_1 with a delay of 1 period.

2. Subgame starting with response of 2: If 2 rejects a proposal, she gets $\delta_2(1 - x_2)$.Thus 2 would accept all proposals that give her at least this amount. Thus for 2's strategy to be optimal given 1's stratgy, we need $\delta_2(1 - x_2) = 1 - x_1$, which implies that $x_1 = 1$.

Thus, the strategy pair is a subgame perfect equilibrium iff $x_1 = 1$.

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