Problem Set 6 Solutions

Exercise 1. Prom Example 1: (Alex$\leftrightarrow$Alice, Bob$\leftrightarrow$Betty),

College Admission Example: $(A \leftrightarrow (\beta,\gamma), B \leftrightarrow \alpha)$; $(A \leftrightarrow (\alpha,\beta), B \leftrightarrow \gamma)$,

Prom Example 2: (Alex$\leftrightarrow$Alice, Bob$\leftrightarrow$Betty); (Alex$\leftrightarrow$Betty, Bob$\leftrightarrow$Alice).

Exercise 2. The men-optimal stable matching and the women-optimal stable matching coincide: $(A \leftrightarrow \gamma, B \leftrightarrow \delta, C \leftrightarrow \alpha, D \leftrightarrow \beta)$; therefore there is a unique stable matching in this example.

Exercise 3. It should have been added to the problem that all men and women are acceptable to each other.

There are at most n^2 rounds because at least one rejection is made in each round. At least one rejection made means at least one proposal made; but in the first round, n proposals are made simultaneously. This reduces the maximum possible number of rounds to $n^2-(n-1)$. In every round before the termination round, at least one woman has not received a proposal; otherwise if every woman has received a proposal, because there are equal numbers of men and women and every person finds every opposite gender acceptable, the process should have terminated.

Therefore, the maximum number of proposals $n(n-1)+1$ involves each of the n men proposing to each of $n-1$ women, and one man proposing to the remaining woman; since n proposals are made in the first step, the maximum number of steps $n(n-1)+1-(n-1)=n^2-2n+2$ involved n proposals made in the first round and 1 proposal made in every subsequent round.

This maximum number of steps can indeed be achieved, with the following preference. Each man m_k, $k=1,\cdots,n-1$ has the following rank:

$$w_k > w_{k+1} > \cdots > w_{n-1} > w_1 > w_2 > \cdots > w_{k-1} > w_n$$

and m_n has the same preference as m_{n-1}

$$w_{n-1} > w_1 > w_2 > \cdots > w_{n-2} > w_n.$$

Women have the following preferences. Woman w_k, $k=1,\cdots,n-1$ has the following preferences

$$m_k < m_{k+1} < \cdots < m_{n-1} < m_n < m_1 < m_2 < \cdots < m_{k-1}$$

Woman w_n can have arbitrary preference. As a result, m_{n-1} is always ranked one above m_n.

When $n = 3$, these preferences are represented by the rank matrix,

$$\begin{array}{cccc} & w_1 & w_2 & w_3 \\ m_1 & 1,3 & 2,1 & 3,1 \\ m_2 & 2,1 & 1,2 & 3,2 \\ m_3 & 2,2 & 1,3 & 3,3 \end{array}$$

You can verify that it takes 5 steps to terminate.

Exercise 4. By the theorem that no man has incentive to misreport under men-proposing algorithms, we only need to check possible profitable deviation by women.

In the Prom Example 1, girls get their favorite boy under truthful reporting (Betty has Bob, Alice has Alex), so they do not have incentive to mis-report.

In the College Admission Example: ($A \leftrightarrow (\beta, \gamma)$, $B \leftrightarrow \alpha$) is the matching from truthful reporting and student-optimal mechanism. B can manipulate by rejecting α (reporting preferences $\gamma \succ \beta \succ \emptyset \succ \alpha \succ \delta$ for example). Then the stable matching resulted is ($A \leftrightarrow (\alpha, \beta)$, $B \leftrightarrow \gamma$).

In the Prom Example 2, either Alice reporting Alex $\succ \emptyset \succ$ Bob, or Betty reporting Bob $\succ \emptyset \succ$ Alex gives Alice and Betty a better outcome (Alex$\leftrightarrow$Alice, Bob$\leftrightarrow$Betty).

Exercise 5 (Osborne 320.1)**.**

The strategic form of the game is:

	PP	PM	MP	MM
RR	1,-1	$\frac{3}{4}+\frac{1}{4}k, -\frac{3}{4}-\frac{1}{4}k$	$\frac{1}{4}-\frac{1}{4}k, -\frac{1}{4}+\frac{1}{4}k$	0,0
RS	$\frac{1}{4}, -\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}k, -\frac{1}{4}-\frac{1}{4}k$	0,0	$\frac{1}{4}k, -\frac{1}{4}k$
SR	$\frac{3}{4}, -\frac{3}{4}$	$\frac{1}{2}, -\frac{1}{2}$	$\frac{1}{4}-\frac{1}{4}k, -\frac{1}{4}+\frac{1}{4}k$	$-\frac{1}{4}k, \frac{1}{4}k$
SS	0,0	0,0	0,0	0,0

First suppose $0 < k < 1$. 1's strategy SS is strictly dominates by mixing RR and RS with probability 0.5. After eliminating SS, 2's strategies PP and PM are strictly dominates by MM. Then we can further eliminate SR by the following argument: 1 plays SR with positive probability only if 2 plays MP with probability 1, but if 2 plays MP, then 1 would place zero probability on RS, making MM optimal. Thus there cannot be any Nash Equilibrium in which 1 places positive probability on SR. The reduced game becomes:

	MP	MM
RR	$\frac{1}{4}-\frac{1}{4}k, -\frac{1}{4}+\frac{1}{4}k$	0,0
RS	0,0	$\frac{1}{4}k, -\frac{1}{4}k$

The unique NE of this game is: 1 plays RR with probability k and 2 plays MP with probability k. So player 1 bluffs with probability k, so the larger k the more likely he is to bluff.

Now consider the case when k>1. The game has 2 pure strategy NE: (RS, MP) and (SS,MP). Also any strategy pair in which 1 assigns positive probability to RS and SS and 2 assigns probability 1 to MP is a Nash equilibrium. In non of these equilibriums 1 bluffs.

Exercise 6 (Osborne 331.1)**.**

The strategic form of the game is (best responses indicated by an asterisks):

If 3 plays L:

	c	d
C	1,1,1*	4*,4*,0
D	3*,3*,2*	3,3*,2*

If 3 plays R:

	c	d
C	1*,1*,1*	0*,0,1*
D	0,0*,0	0*,0*,0

The are 2 pure strategy Nash equilibria: (D,c,L) and (C,c,R).

Consider the equilibrium (D,c,L). 2's action is not sequentially rational because given the other players' strategies, he is better off choosing d when it is his time to move. Therefore there is no weak sequential equilibrium in which this is the strategy profile.

Now consider the equilibrium (C,c,R). The actions of 1 and 2 are sequentially rational. Player 3's information set is not reached, so we can specify any beliefs there. If 3 believes that the history is D with probability at most $\frac{1}{3}$, choosing R is optimal. Therefore the game has weak sequential equilibria in which (C,c,R) is the strategy profile and 3's beliefs assigns probability of at most $\frac{1}{3}$ to D.

Exercise 7 (Osborne 335.1)**.**

Label the payoffs at (Weak, Ready, A) as (a_1, a_2) and those at (Weak, Ready, F) as (b_1, b_2). Denote the information set where challenger is Strong as 1.1 and where he is Weak as 1.2, the information set of the incumbent after history (Strong, Ready) as 2.4 and that of the incumbent after history (Weak, Unready) as 2.3.

- Separating equilibria
  - The following is a separating equilibrium if $a_1 \leq 3$: the challenger chooses Ready when Strong and Unready when Weak. Given this, the incumbent assigns probability 1 to the history (Strong, Ready) and probability 1 to the history (Weak, Unready). Given these beliefs the incumbent chooses A at information set 2.4 and F at information set 2.3.
- Pooling equilibria - here is one of the equilibria

- Call the belief that the incumbent places on the history (Strong, Ready) β and his belief on history (Strong, Unready) α.
- If both types of player Challenger choose U, then $\alpha = p$ and β is unrestricted. Thus the incumbent's actions at information set 2.3 is F is $p < \frac{1}{4}$, A if $p > \frac{1}{4}$ and mix between A and F if $p = \frac{1}{4}$.
- now consider information set 2.4. If $a_2 > b_2$ then incumbent will always play A since it strictly dominates playing F.
- If $a_2 \leq b_2$ consider following cases:
  * If $p < \frac{1}{4}$ then incumbent chooses F at 2.3. Let q denote the probability of of playing A at information set 2.4. Then sequential rationality for the challenger requires that $q \leq \frac{1}{2}$ and $a_1 q + b_1(1-q) \leq 3$.

Exercise 8 (Osborne 335.2)**.** Separating equilibrium

- The parent's strategy is sequentially rational (SR) if keep when quiet,

$$1 + rV \geq S + r,$$

  and give when squawk,

$$S + r(1-t) \geq 1$$

- The offspring's strategy is SR if squawk when hungry

$$1 - t + rS \geq r$$

  and quiet when not hungry,

$$V + r \geq 1 - t + rS$$

- Combining these conditions we get:

$$\frac{1-S}{1-t} \leq r \leq \frac{1-S}{1-V}$$

  and

$$\frac{1-V-t}{1-S} \leq r \leq \frac{1-t}{1-S}$$

- The condition $r < \frac{1-V}{1-S}$ is consistent with the LHS of the second inequality only if $t > 0$.

Pooling equilibrium

- let p be the probability that the parent believes the offspring is hungry when it is quiet.
- For the parent's strategy to be sequentially rational (SR) we need:

$$p+(1-p)(1+rV)\geq S+r$$

  and

$$1+rV(1-t)\geq S+r(1-t)$$

- The parent's behavior does not depend on the offspring's action. If $r<\frac{1-S}{1-(1-p)V}$ then both of these conditions are satisfied.

Exercise 9 (Spence Job Market Signaling)**.** We have a separating equilibrium in which the high ability person chooses to pursue a degree and the low ability person chooses not to, and the firm hires any person with a degree as an executive and anyone not having a degree as a worker.

We also have a pooling equilibrium in which everyone chooses not to pursue a degree and the firm chooses W for anyone with a degree and without a degree. The belief p the firm has is off equilibrium path. As long as $5p-5(1-p)\leq 2p+3(1-p)\Rightarrow p\leq 8/11$, it constitutes a WSE.

There is no equilibrium in which the student plays a mixed behavioral strategy. The low ability person is always better off not getting a degree because the cost is too high. Suppose a student of high ability gets a degree with probability d. He is indifferent between getting a degree and not getting when

$$2t=3v+1-v=2v+1\Rightarrow t=v+0.5,$$

that is $0\leq v\leq 0.5$. Suppose $v>0$. in order for the firm to be indifferent between E and W when a person has no degree, we need that

$$5q-5(1-q)=2q+3(1-q)\Rightarrow q=8/11.$$

However, since low ability person always plays No, $q\leq 1/2$ (by Bayes' Rule, if a high ability person's probability of playing is r, then $q=0.5r/(0.5r+0.5)=r/(1+r)<1/2$ when $r<1$). If $v=0$, then $t=0.5$, but when $t=0.5$, a high ability person is not