# Math Appendix and Pset 1 Question 5 

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Friday, January 20, $2012^{\dagger}$

## 1 Calculus Formulas

Theorem 1. (Chain Rule)

$$
\frac{\partial f(g(x))}{\partial x}=\frac{\partial f(g)}{\partial g} \cdot \frac{\partial g(x)}{\partial x}
$$

Corollary 2. (Product Rule)

$$
(f(x) \cdot g(x))^{\prime}=f^{\prime}(x) g(x)+f(x) g^{\prime}(x) .
$$

Remark. Why is product rule a corollary of chain rule? $d(f g)=d(f g) / d f \cdot d f / d x+$ $d(f g) / d g \cdot d g / d x=g \cdot d f / d x+f \cdot d g / d x$.

And some corollaries of the product rule:

## Corollary 3. (Quotient Rule)

$$
\left(\frac{f(x)}{g(x)}\right)^{\prime}=\frac{g(x) f^{\prime}(x)-f(x) g^{\prime}(x)}{g(x)^{2}} .
$$

## (Integration By Parts)

$$
\int v \cdot d u=u \cdot v-\int u \cdot d v .
$$

Remark. Quotien Rule: let $g=1 / g$. Why is integration by parts a corollary of product rule? Informally, $d(u v) / d x=d u / d x \cdot v+u \cdot d v / d x$, put an integral sign, $\int d(u v)=$ $\int u d v+\int v d u$, rearrange, we get the integration by parts.

Here is an important one that captures all variants of differentiation involving the integral sign:

Theorem 4. (Differentiation under the integration sign) Suppose $a(\cdot)$ and $b(\cdot)$ are functions of $x, f(\cdot)$ is a function of $x$ and $t$, and $F$ is a function of $x$ taking the following form:

$$
F(x)=\int_{a(x)}^{b(x)} f(x, t) d t
$$

[^0]then the derivative of $F(x)$ with respect to $x$ is
\[

$$
\begin{equation*}
\frac{\partial F(x)}{\partial x}=f(x, b(x)) b^{\prime}(x)-f(x, a(x)) a^{\prime}(x)+\int_{a(x)}^{b(x)} \frac{\partial f(x, t)}{\partial x} d t \tag{1}
\end{equation*}
$$

\]

Corollary 5. (Second Fundamental Theorem of Calculus) Let $f(x)=F^{\prime}(x)$, i.e. $f$ is derivative of $F$ or $F$ is anti-derivative of $f$. If $f$ is integrable, then

$$
\int_{a}^{b} f(x) d x=F(b)-F(a),
$$

or

$$
\frac{\partial}{\partial x} \int_{0}^{x} f(t) d t=f(x) .
$$

Proof. Basically set $a(x)=0, b(x)=x, f(x, t)=f(t)$, so that right hand side of Equation 1 becomes $f(x) \cdot 1-f(0) \cdot 0-\int_{0}^{x} 0 d t=f(x)$.

Corollary 6. (Leibniz integral rule) Let $f$ be a function of $x$ and $y$, then

$$
\frac{d}{d x} \int_{y_{0}}^{y_{1}} f(x, y) d y=\int_{y_{0}}^{y_{1}} \frac{\partial}{\partial x} f(x, y) d y .
$$

Proof. Set $a=b=$ constant in Equation 1 .
Theorem. (Exchange of Integrals): Find a math textbook for detailed treatment, for our purpose, knowing the following is sufficient for current problem set:

$$
\int_{0}^{1} \int_{0}^{v} f(v, x) d x d v=\int_{0}^{1} \int_{x}^{1} f(v, x) d v d x
$$

## 2 Order Statistics

The $k^{\text {th }}$ order statistic of a sample refers to its $k$-th largest value. It is important in auction theory because of first-price and second-price auctions that have payments dependent of the order statistics. Here shows some derivations of the order statistics. Suppose there are $N$ bidders, and let each value $v_{i}$ drawn from common distribution $F$, then let $p_{k}$ denote the density of the $k$-th largest value being $v$. First, the density of the largest value is calculated.

$$
\begin{aligned}
p_{1}(v) & =\operatorname{Pr}(v \text { is largest }) \\
& =\sum_{i=1}^{N} \operatorname{Pr}\left(v_{i} \text { is max } \mid v_{i}=v\right) \operatorname{Pr}\left(v_{i}=v\right) \\
& =\sum_{i=1}^{N} \operatorname{Pr}\left(v_{i}=v\right) \cdot \operatorname{Pr}\left(v_{j} \leq v \forall j \neq i\right) \\
& =\sum_{i=1}^{N}\left[f(v) \cdot \prod_{j \neq 1} \operatorname{Pr}\left(v_{j} \leq v\right)\right] \\
& =N f(v) F^{N-1}(v)
\end{aligned}
$$

CDF is just by integrating,

$$
P_{1}(v)=\int_{0}^{v} p_{1}(x) d x=\int_{0}^{v} N F^{N-1}(x) f(x) d x=F^{N}(v)
$$

easily we interpret it as that the probability that every value is smaller than $v$. Similarly, density of second largest value is calculated.

$$
\begin{aligned}
p_{2}(v) & =\operatorname{Pr}(v \text { is 2nd largest }) \\
& =\sum_{i=1}^{N} \operatorname{Pr}\left(v_{i}=v\right) \cdot \operatorname{Pr}\left(v_{i} \text { is 2nd largest } \mid v_{i}=v\right) \\
& =\sum_{i=1}^{N} f(v) \cdot \operatorname{Pr}\left(v<v_{k}, v \geq v_{j} \text { for } k \neq i \text { and } \forall j \neq k, i\right) \\
& =\sum_{i=1}^{N} f(v) \cdot \sum_{k \neq i} \operatorname{Pr}\left(v<v_{k}\right) \operatorname{Pr}\left(v_{j} \leq v \forall j \neq k, i\right) \\
& =\sum_{i=1}^{N} f(v) \cdot \sum_{k \neq i} \operatorname{Pr}\left(v_{k}>v\right) \cdot F^{N-2}(v) \\
& =\sum_{i=1}^{N} f(v) \sum_{k \neq i}(1-F(v)) \cdot F^{N-2}(v) \\
& =N \cdot f(v) \cdot(N-1) \cdot(1-F(v)) \cdot F^{N-2}(v)
\end{aligned}
$$

CDF is similarly calculated:

$$
\begin{aligned}
P_{2}(v) & =\int_{0}^{v} N \cdot f(x) \cdot(N-1) \cdot(1-F(x)) \cdot F^{N-2}(x) d x \\
& =N(N-1)\left[\int_{0}^{v} F^{N-2}(x) f(x) d x-\int_{0}^{v} F^{N-1}(x) f(x) d x\right] \\
& =N(N-1)\left[\frac{F^{N-1}(v)}{N-1}-\frac{F^{N}(v)}{N}\right] \\
& =N F^{N-1}(v)-(N-1) F^{N}(v) \\
& =F^{N-1}(v)[N-(N-1) F(v)]
\end{aligned}
$$

Now patten of reasoning is observed, and density of $k$-th largest value being $v$ is calculated.

$$
\begin{aligned}
p_{k}(v) & =\sum_{i=1}^{N} \operatorname{Pr}\left(v_{i}=v\right) \operatorname{Pr}\left(v_{i} \text { is k-th largest } \mid v_{i}=v\right) \\
& =N \cdot f(v) \cdot\binom{N-1}{k-1} \cdot F^{N-1-(k-1)}(v) \cdot(1-F(v))^{k-1} \\
& =N \cdot\binom{N-1}{k-1} \cdot f(v) \cdot(F(v))^{N-k} \cdot(1-F(v))^{k-1} \\
& =k \cdot\binom{N}{k} \cdot f(v) \cdot(F(v))^{N-k} \cdot(1-F(v))^{k-1} .
\end{aligned}
$$

With these fundamental mathematical techniques, we can tackle any auctions problem in the first half of the quarter:

## 3 Probem Set 1 Question 5

Question. Suppose there are just two bidders. In a second-price, all-pay auction, the two bidders simultaneously submit sealed bids. The highest bid wins the object and both bidders pay the second-highest bid.
(a) Find the unique symmetric equilibrium bidding function. Interpret.

ANSWER: The bidding function $b$ is assumed to be strictly increasing. Now suppose the value of the other bidder is $x$, then utility of the bidder when he bids according to $b(r)$ if his value is $v$ and opponent's value is $p$ is

$$
u(r \mid v, p)= \begin{cases}v-b(x) & : r>x \\ -b(r) & : r<x\end{cases}
$$

Then the expected utility of bidding according $b(r)$ when value is $v$ is

$$
\begin{aligned}
u(r \mid v) & =\int_{0}^{1}\left[1_{r>x} \cdot(v-b(x))+1_{r<x}(-b(r))\right] d F(x) \\
& =\int_{0}^{r}(v-b(p)) d F(x)+\int_{r}^{1}(-b(r)) d F(x) \\
& =\int_{0}^{r}(v-b(x)) f(x) d x-(1-F(r)) b(r) \\
& =v F(r)-(1-F(r)) b(r)-\int_{0}^{r} b(x) f(x) d x
\end{aligned}
$$

foc is

$$
\begin{aligned}
\frac{\partial u(r \mid v)}{\partial r} & =v f(r)-(1-F(r)) b^{\prime}(r)+f(r) b(r)-f(r) b(r) \\
& =v f(r)-(1-F(r)) b^{\prime}(r)
\end{aligned}
$$

and it equals 0 at $r=v$, so

$$
b^{\prime}(v)=\frac{v f(v)}{1-F(v)} \forall v .
$$

That is,

$$
b(v)=\int_{0}^{v} \frac{x f(x)}{1-F(x)} d x+k=\int_{0}^{v} \frac{x f(x)}{1-F(x)} d x .
$$

since $b(0)=0$. This symmetric equilibrium bidding function can be interpreted as the opponent's expected value conditional on the bidder wins the object (bidder's value greater than the opponent's).
In order to show that this bidding function is indeed an equilibrium bidding function,
we need to show that the utility is indeed maximized at $r=v$,

$$
\begin{aligned}
\frac{\partial u(r \mid v)}{\partial r} & =v f(r)-(1-F(r)) b^{\prime}(r) \\
\frac{\partial^{2} u(r \mid v)}{\partial r^{2}} & =v f^{\prime}(r)-(-f(r)) b^{\prime}(r)-(1-F(r)) b^{\prime \prime}(r) \\
& =v f^{\prime}(r)-(-f(r)) \frac{r \cdot f(r)}{1-F(r)}-(1-F(r)) \frac{(1-F(r))(r \cdot f(r))^{\prime}-r f(r)(-f(r))}{(1-F(r))^{2}} \\
\left.\right|_{r=v} & =v f^{\prime}(v)+\frac{v f^{2}(v)}{1-F(v)}-f(v)-v f^{\prime}(v)-\frac{v f^{2}(v)}{1-F(v)} \\
& =-f(v)<0
\end{aligned}
$$

And we need to check that $u(r \mid v)>0$, and it indeed is so:

$$
\begin{aligned}
u(v \mid v) & =v F(v)-(1-F(v)) b(v)-\int_{0}^{v} b(x) f(x) d x \\
& =v F(v)-(1-F(v)) b(v)-\left[b(v) F(v)-\int_{0}^{v} F(x) b^{\prime}(x) d x\right] \\
& =v F(v)-(1-F(v)) b(v)-\left[b(v) F(v)-\int_{0}^{v} F(x) \frac{x f(x)}{1-F(x)} d x\right] \\
& =v F(v)-b(v)+\int_{0}^{v} F(x) \frac{x f(x)}{1-F(x)} d x \\
& =v F(v)-b(v)-\int_{0}^{v} \frac{x(1-F(x))}{1-F(x)} f(x) d x+\int_{0}^{v} \frac{x}{1-F(x)} f(x) d x \\
& =v F(v)-\int_{0}^{v} x f(x) d x \\
& =\int_{0}^{v}(v-x) f(x) d x>0 .
\end{aligned}
$$

(b) Do bidders bid higher or lower than in a first-price, all-pay auction?

ANSWER: In all-pay first-price auction with two players, the equilibrium bidding function can be simplified (by "reverse" integration by part) to be

$$
\begin{aligned}
\hat{b}^{\mathrm{APFPA}}(v) & =F(v)\left[v-\int_{0}^{v} \frac{F(x)}{F(v)} d x\right]=F(v) v-\int_{0}^{v} F(x) d x \\
& =\int_{0}^{v} x d F(x)=\int_{0}^{v} x f(x) d x
\end{aligned}
$$

On the other hand,

$$
\hat{b}^{\mathrm{APSPA}}(v)=\int_{0}^{v} \frac{x f(x)}{1-F(x)} d x
$$

Since $1-F(x)<1$ for all $x>0, \hat{b}^{\text {APSPA }}(v)>\hat{b}^{\text {APFPA }}(v)$. This is true because in the second-price auction, bidders pay lower than their bids when they win, so they have greater incentives to boost their bids.
Remark. ${ }^{1}$ However, the relationship between $\hat{b}$ APSPA $(v)$ and $\hat{b}^{\text {SPA }}(v)$ is indeterminable.

[^1]For example, if $F(x)=x$, then

$$
\begin{aligned}
\hat{b}^{\mathrm{APSPA}}(v)-\hat{b}^{\mathrm{SPA}}(v) & =\int_{0}^{v}\left(\frac{x}{1-x}-1\right) d x=\int_{0}^{v}\left(\frac{1}{1-x}-\frac{1-x}{1-x}-1\right) d x \\
& =-\left.\ln (1-x)\right|_{0} ^{v}-2 v=-\ln (1-v)-2 v \\
& =-\ln \left[(1-v) e^{2 v}\right]
\end{aligned}
$$

when $v>0.797, \hat{b}^{\text {APSPA }}(v)>\hat{b}^{\text {SPA }}(v)$, when $v<0.797, \hat{b}^{\text {APSPA }}(v)<\hat{b}^{\text {SPA }}(v)$.
Take another example: If the value is exponentially distributed, $F(x)=1-\exp (-\lambda x)$, $f(x)=\lambda \exp (-\lambda x)$, so $\frac{f(x)}{1-F(x)}=\lambda$.

$$
\hat{b}^{\mathrm{APSPA}}(v)-\hat{b}^{\mathrm{SPA}}(v)=\int_{0}^{v}(\lambda x-1) d x=\frac{\lambda}{2} v^{2}-v=v\left(\frac{\lambda v}{2}-1\right) .
$$

For $v<2 / \lambda, \hat{b}^{\text {APSPA }}(v)<\hat{b}^{\text {SPA }}(v)$; for $v>2 / \lambda, \hat{b}^{\text {APSPA }}(v)>\hat{b}^{\text {SPA }}(v)$ for $v \in[0,1]$.
The incentive to bid above one's own value is due to the combination of all-pay nature of the auction and the relative low winning payment. Because only the winner pays lower than his bid, there is a bigger incentive to win. Probability of winning is bigger when the bidder bids higher, and when bidder's value is high, he has higher incentive to guarantee himself winning so that he does not pay his bid for getting nothing. Therefore, as we see from previous numeric examples, for higher $v$, bidder's bid increases to higher than $v$.
Take a concrete scenario. Suppose $v<p<v+\epsilon$, i.e. value of bidder is smaller than the bid of the opposing bidder, but not much smaller, then if he bids $v$ (or anything below $p$ ), he pays his bid $v$ and gets nothing, so his utility is $-v$. However, if he raises his bid to $v+\epsilon$, then he pays $p$ and gets $v$ utility from obtaining the object, so his net utility is $v-p>-\epsilon$. As $v$ increases, his disutility of losing is getting bigger for bidding $v$, therefore he has more and more incentive to bid higher than his value as his value for the object increases.
(c) Find an expression for the seller's expected revenue.

ANSWER: The seller's expected revenue is the sum of the bids for all agents since everyone pays, and by change of integration signs,

$$
\begin{aligned}
R^{\text {APSPA }} & =2 \int_{0}^{1} 2(1-F(v)) f(v) \hat{b}^{\text {APSPA }}(v) d v \\
& =4 \int_{0}^{1}(1-F(v)) f(v)\left[\int_{0}^{v} \frac{x f(x)}{1-F(x)} d x\right] d v \\
& =4 \int_{0}^{1} \int_{0}^{v}(1-F(v)) f(v)\left[\frac{x f(x)}{1-F(x)}\right] d x d v \\
& =4 \int_{0}^{1} \int_{x}^{1}(1-F(v)) f(v)\left[\frac{x f(x)}{1-F(x)}\right] d v d x \\
& \left.=-\left.2 \int_{0}^{1}(1-F(v))^{2}\right|_{x} ^{1} \cdot \frac{x f(x)}{1-F(x)} d x \quad \text { (because } F(1)=1\right) \\
& =2 \int_{0}^{1} x f(x)(1-F(x)) d x
\end{aligned}
$$

Or by integration by parts,

$$
\begin{aligned}
R^{\mathrm{APSPA}} & =4 \int_{0}^{1}(1-F(v)) f(v) b(v) d v \\
& =-4 \cdot \frac{1}{2}\left[\left.(1-F(v))^{2} b(v)\right|_{0} ^{1}-\int_{0}^{1}(1-F(v))^{2} b^{\prime}(v) d v\right] \\
& =-2\left[(1-F(1))^{2} b(1)-(1-F(0))^{2} b(0)-\int_{0}^{1}(1-F(v))^{2} \frac{v f(v)}{1-F(v)} d v\right] \\
& =2 \int_{0}^{1}(1-F(v)) v f(v) d v
\end{aligned}
$$

(d) Both with and without revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.
ANSWER: Without revenue equivalence theorem,

$$
R^{\mathrm{SPA}}=2 \int_{0}^{1} x f(x)(1-F(x)) d x=R^{\mathrm{APSPA}}
$$

as shown in part (c). Since both SPA and APSPA assign the object to the bidder with the highest value, and are both incentive-compatible with bidder with value zero indifferent between the mechanism as their expected payoff is zero, by RET, the two mechanisms generate the same expected revenues.


[^0]:    *hanzhe@uchicago.edu. All errors are solely mine.
    ${ }^{\dagger}$ Updated 01:00, 01/31/2012.

[^1]:    ${ }^{1}$ Thank Taesup Lee for pointing out the previous mistake.

