ECON244 Midterm Solution

November 27, 2012

In solving these problems, I ask you to use the utility function u as explicitly as possible and to rely on little as possible on the general proofs that tell you some of the relationships hold, you may wish, however, adapt these proofs for the special utility function at hand.

Let $u(\mathbf{x}) = \ln(x_1+2) + 2\ln(x_2+3) + 4\ln(x_3+2)$ and $p_1^* = 2$, $p_2^* = 3$, $p_3^* = 1$ and $\omega^* = 27$.

1. Compute demand $x(\mathbf{p}^*, \omega^*)$.

Solution. *Utility maximization problem is*

$$\begin{array}{ll} \max_{\mathbf{x}} & \ln(x_1+2)+2\ln(x_2+3)+4\ln(x_3+2) \\ s.t. & p_1^*x_1 + p_2^*x_2 + p_3^*x_3 \le \omega^* \end{array}$$

Lagrangian is

$$\mathscr{L} = \ln(x_1+2) + 2\ln(x_2+3) + 4\ln(x_3+2) + \lambda\left(\omega - p_1^*x_1 - p_2^*x_2 - p_3^*x_3\right).$$

FOCs are

$$\begin{aligned} \frac{1}{x_1+2} &= \lambda p_1^* \Rightarrow x_1 = \frac{1}{\lambda p_1^*} - 2 \\ \frac{2}{x_2+3} &= \lambda p_2^* \Rightarrow x_2 = \frac{2}{\lambda p_2^*} - 3 \\ \frac{4}{x_3+2} &= \lambda p_3^* \Rightarrow x_3 = \frac{4}{\lambda p_3^*} - 2 \\ p_1^*x_1 + p_2^*x_2 + p_3^*x_3 &= \omega^* \end{aligned}$$

Plugging in, we

$$\begin{aligned} \left(\frac{1}{\lambda} - 2p_1^*\right) + \left(\frac{2}{\lambda} - 3p_2^*\right) + \left(\frac{4}{\lambda} - 2p_3^*\right) &= \omega^* \\ \omega^* + 2p_1^* + 3p_2^* + 2p_3^* &= \frac{7}{\lambda}. \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= \frac{1}{7}\left(\omega^* + 2p_1^* + 3p_2^* + 2p_3^*\right)\frac{1}{p_1^*} - 2 = \frac{1}{7}\left(\omega^* + 3p_2^* + 2p_3^*\right)\frac{1}{p_1^*} - \frac{10}{7} \\ x_2 &= \frac{2}{7}\left(\omega^* + 2p_1^* + 3p_2^* + 2p_3^*\right)\frac{1}{p_2^*} - 3 = \frac{2}{7}\left(\omega^* + 2p_1^* + 2p_3^*\right)\frac{1}{p_2^*} - \frac{3}{7} \\ x_3 &= \frac{4}{7}\left(\omega^* + 2p_1^* + 3p_2^* + 2p_3^*\right)\frac{1}{p_3^*} - 2 = \frac{4}{7}\left(\omega^* + 2p_1^* + 3p_2^*\right)\frac{1}{p_3^*} - \frac{1}{7} \end{aligned}$$

For the particular values, we get that $1/\lambda = 6$, *and* $x_1 = 1$, $x_2 = 1$, $x_3 = 22$.

2. Compute the terms $\frac{\partial x_h}{\partial p_k} + x_k \frac{\partial x_h}{\partial \omega}$ evaluated at $(\mathbf{p}^*, \omega^*)$.

Solution. *$\partial x_h / \partial p_i$ and $\partial x_h / \partial \omega$ are as below,*

h	1	2	3
$\frac{\partial x_h}{\partial p_1}$	$-\frac{1}{7}\frac{1}{p_1^{*2}}(\omega + 3p_2^* + 2p_3^*)$	$\frac{2}{7} \cdot \frac{2}{p_2^*}$	$\frac{4}{7} \cdot \frac{2}{p_3^*}$
$\frac{\partial x_h}{\partial p_2}$	$\frac{1}{7} \cdot \frac{3}{p_1^*}$	$-\frac{2}{7}\frac{1}{p_2^2}(\omega + 2p_1^* + 2p_3^*)$	$\frac{4}{7} \cdot \frac{3}{p_3^*}$
$\frac{\partial x_h}{\partial p_3}$	$\frac{1}{7} \cdot \frac{2}{p_1^*}$	$\frac{2}{7} \cdot \frac{2}{p_2^*}$	$-\frac{4}{7}\frac{1}{p_3^2}(\omega + 2p_1^* + 3p_2^*)$
$\frac{\partial x_h}{\partial \omega}$	$\frac{1}{7} \cdot \frac{1}{p_1^*}$	$\frac{2}{7} \cdot \frac{1}{p_2^*}$	$\frac{4}{7} \cdot \frac{1}{p_3^*}$

then the matrix is

$$\begin{array}{ccc} -\frac{1}{7}\frac{1}{p_1^{*2}}(\omega + 3p_2^* + 2p_3^*) + x_1 \frac{1}{7} \cdot \frac{1}{p_1^*} & \frac{2}{7} \cdot \frac{2}{p_2^*} + x_1 \frac{2}{7} \cdot \frac{1}{p_2^*} & \frac{4}{7} \cdot \frac{2}{p_3^*} + x_1 \frac{4}{7} \cdot \frac{1}{p_3^*} \\ \frac{1}{7} \cdot \frac{3}{p_1^*} + x_2 \frac{1}{7} \cdot \frac{1}{p_1^*} & -\frac{2}{7}\frac{1}{p_2^2}(\omega + 2p_1^* + 2p_3^*) + x_2 \frac{2}{7} \cdot \frac{1}{p_2^*} & \frac{4}{7} \cdot \frac{3}{p_3^*} + x_2 \frac{4}{7} \cdot \frac{3}{p_3^*} \\ \frac{1}{7} \cdot \frac{2}{p_1^*} + x_3 \frac{1}{7} \cdot \frac{1}{p_1^*} & \frac{2}{7} \cdot \frac{2}{p_2^*} + x_3 \frac{2}{7} \cdot \frac{1}{p_2^*} & -\frac{4}{7}\frac{1}{p_3^2}(\omega + 2p_1^* + 3p_2^*) + x_3 \frac{4}{7} \cdot \frac{1}{p_3^*} \end{array}$$

Plugging in $(\mathbf{p}^*, \omega^*)$, the matrix is

$$\frac{1}{7} \times \begin{array}{ccc} -9 & 2 & 12 \\ 2 & -\frac{20}{3} & 16 \\ 12 & 16 & -72 \end{array}$$

3. Verify that the matrix of these terms is symmetric and negative semi-definite.

Solution. *It is symmetric as its transpose is the same matrix.*
Now we need show that it is NSD. All the diangonal terms are negative. The determinant is negative. Any two by two matrices has positive determinant. By Proposition 9-2) of ConcavityNSD.pdf, the matrix is NSD.

4. Compute $v(p^*, \omega^*)$ at this (p^*, ω^*).

Solution. *The indirect utility is*

$$\begin{aligned} v(p^*, \omega^*) &= \ln(x_1 + 2) + 2\ln(x_2 + 3) + 4\ln(x_3 + 2) \\ &= \ln(3) + 2\ln(4) + 4\ln(24) = \ln\left(3 \times 4^2 \times 24^4\right) = \ln\left(2^{16} \times 3^5\right) \end{aligned}$$

5. Derive $H_h(\mathbf{p}^*, u)$, the Hicksian compensated demand for the h^{th} commodity associated with the utility function u when the utility level is $v(\mathbf{p}^*, w^*)$. Verify that $\partial H_h / \partial p_h \le 0$.

Solution. *Let $u = v(\mathbf{p}^*, \omega^*)$, the expenditure minimization problem is*

$$\begin{array}{ll} \min_{\mathbf{H}} & p_1^* H_1 + p_2^* H_2 + p_3^* H_3 \\ s.t. & \ln(H_1 + 2) + 2\ln(H_2 + 3) + 4\ln(H_3 + 2) \ge u \end{array}$$

Lagrangian is

$$\mathscr{L} = p_1^* H_1 + p_2^* H_2 + p_3^* H_3 + \lambda\left[u - \ln(H_1 + 2) - 2\ln(H_2 + 3) - 4\ln(H_3 + 2)\right]$$

FOCs derive

$$H_1 = \frac{1}{\lambda p_1^*} - 2, H_2 = \frac{2}{\lambda p_2^*} - 3, H_3 = \frac{4}{\lambda p_3^*} - 2$$

and

$$\ln(H_1+2) + 2\ln(H_2+3) + 4\ln(H_3+2) = u$$

Substitute in,

$$\exp(u)\, p_1^* \left(\frac{p_2^*}{2}\right)^2 \left(\frac{p_3^*}{4}\right)^4 = \left(\frac{1}{\lambda}\right)^7.$$

Then the Hicksian demand is

$$\begin{aligned}
H_1 &= \exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}}\frac{1}{p_1^*} - 2 \\
H_2 &= \exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}}\frac{2}{p_2^*} - 3 \\
H_3 &= \exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}}\frac{4}{p_3^*} - 2
\end{aligned}$$

Take the derivative, we get

$$\begin{aligned}
\frac{\partial H_1}{\partial p_1^*} &= -\frac{6}{7}\exp\left(\frac{u}{7}\right)(p_1^*)^{-\frac{6}{7}-1}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}} < 0 \\
\frac{\partial H_2}{\partial p_2^*} &= -\frac{5}{7}\frac{1}{2}\exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{-\frac{5}{7}-1}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}} < 0 \\
\frac{\partial H_3}{\partial p_3^*} &= -\frac{3}{7}\frac{1}{4}\exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{-\frac{3}{7}-1} < 0
\end{aligned}$$

6. Derive the function $e(\mathbf{p}, v(\mathbf{p}^*, \omega^*))$ and verify that it satisfies the monotonicity and concavity (of prices) properties of an expenditure function.

Solution. *The expenditure function is*

$$e(\mathbf{p}^*, u) = \mathbf{p} \cdot \mathbf{H}(\mathbf{p}^*, u) = 7\exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}} - 2p_1^* - 3p_2^* - 2p_3^*.$$

It is monotonic in u, as

$$\frac{\partial e(\mathbf{p}^*, u)}{\partial u} = \exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}} > 0.$$

And it is monotonice in $\mathbf{p}$*, as shown by part 7 that* $\partial e/\partial p_h = H_h \geq 0$*. In order to show that it is concave in* $\mathbf{p}$*, by Proposition 8 of ConcavityNSD.pdf, we need to show that its Hessian is NSD.*

7. Verify that $\nabla e(\mathbf{p}^*, u) = \mathbf{H}$, where $u = v(\mathbf{p}^*, \omega^*)$.

Solution. *We can check directly by differentiating e with respect to* **p***,*

$$\begin{aligned}
\frac{\partial e}{\partial p_1} &= 7\frac{1}{7}\exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}-1}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}}-2=\frac{1}{\lambda}\frac{1}{p_1^*}-2=H_1 \\
\frac{\partial e}{\partial p_2} &= 7\frac{2}{7}\frac{1}{2}\exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}-1}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}}-3=\frac{1}{\lambda}\frac{2}{p_2^*}-3=H_2 \\
\frac{\partial e}{\partial p_3} &= 7\frac{4}{7}\frac{1}{4}\exp\left(\frac{u}{7}\right)(p_1^*)^{\frac{1}{7}}\left(\frac{p_2^*}{2}\right)^{\frac{2}{7}}\left(\frac{p_3^*}{4}\right)^{\frac{4}{7}-1}-2=\frac{1}{\lambda}\frac{4}{p_3^*}-2=H_3
\end{aligned}$$

8. Derive $S_h(\mathbf{p},x(\mathbf{p}^*,\omega^*))$, the Slutsky compensated demand for the h^{th} commodity associated with the demand function $x(\cdot)$ generated by the utility function u and verify that $\partial S_h/\partial p_h \le 0$.

Solution. *The Slutsky compensated demand is derived from utility maximization problem:*

$$\max_{\mathbf{S}} u(S) \quad s.t.\ p_1^*S_1+p_2^*S_2+p_3^*S_3 < p_1^*x_1+p_2^*x_2+p_3^*x_3$$

and Lagrangian is

$$\mathscr{L}=\ln(S_1+2)+2\ln(S_2+3)+4\ln(S_3+2)+\lambda\left[(p_1^*+p_2^*+22p_3^*)-p_1^*S_1-p_2^*S_2-p_3^*S_3\right]$$

so that the FOCs are

$$\begin{aligned}
\frac{1}{S_1+1} &= \lambda p_1^* \\
\frac{2}{S_2+3} &= \lambda p_2^* \\
\frac{4}{S_3+2} &= \lambda p_3^* \\
p_1^*+p_2^*+22p_3^* &= p_1^*S_1+p_2^*S_2+p_3^*S_3
\end{aligned}$$

since FOCs are the same as in part 1, the Slutsky demand is

$$\begin{aligned}
x_1 &= \frac{1}{7}(p_1^*+p_2^*+22p_3^*+2p_1^*+3p_2^*+2p_3^*)\frac{1}{p_1^*}-2 \\
&= \frac{1}{7}(4p_2^*+24p_3^*)\frac{1}{p_1^*}-\frac{11}{7} \\
x_2 &= \frac{2}{7}(3p_1^*+24p_3^*)\frac{1}{p_2^*}-\frac{13}{7} \\
x_3 &= \frac{4}{7}(p_1^*+p_2^*+22p_3^*+2p_1^*+3p_2^*+2p_3^*)\frac{1}{p_3^*}-2 \\
&= \frac{4}{7}(3p_1^*+4p_2^*)\frac{1}{p_3^*}+4
\end{aligned}$$

9. Verify that $\frac{\partial^2 e}{\partial p_h \partial p_k}(\mathbf{p}^*,v(\mathbf{p}^*,\omega^*))$ equal the terms derived in 2.

Solution. *From 7, we know that $\partial e/\partial p_h = H_h$, since e is continuously differentiable,*

$$\frac{\partial^2 e}{\partial p_h \partial p_k}=\frac{\partial}{\partial p_h}\left(\frac{\partial e}{\partial p_k}\right)=\frac{\partial}{\partial p_k}\left(\frac{\partial e}{\partial p_h}\right),$$

the matrix of the terms is symmetric.

Kihlstrom

Physician Dr. D maximizes utility, which is an increasing function of leisure L and income I. He earns income from providing medical services y for which he is paid p dollars per unit of y. He produces medical services by combining his labor services $M$with the services of medial aides acording to

$$y = f(M,x).$$

Assume that the market for medical aides and medical services are competitive, with w denoting the price of the services of aides. Assume further that Dr. D has T hours to divide between L and M.

1. Show that the change in the demand for aides, associated with a rise in their cost w can be decomposed into an income and a substitution effect, with the latter effect always negative.

Solution. *First assume that utility and production functions are increasing, quasiconcave and continuously differentiable (in particular, $u_I, u_L, f_M, f_x > 0$, $u_{LL}, u_{II}, f_{xx}, f_{MM} < 0$). And assume that leisure is normal good.*
The constrained maximization problem is

$$\begin{array}{ll} \max_{L,I} & u(L,I) \\ s.t. & L+M=T \\ & I = pf(M,x) - wx. \end{array}$$

Substituting the constraints in, we get the following (unconstrained) maximization problem,

$$\max_{M,x} u(T-M, pf(M,x) - wx).$$

FOCs are

$$\begin{array}{rrcl} [M] & u_L(L,I) & = & pf_M(M,x)\,u_I(L,I) \\ [x] & pf_x(M,x) & = & w \end{array}$$

By implicit function theorem, now differentiate both sides by w, the two FOCs become

$$-u_{LL}\frac{\partial M}{\partial w} + u_{LI}\frac{\partial I}{\partial w} = p\frac{\partial f_M}{\partial w}u_I + pf_M\left(-u_{IL}\frac{\partial M}{\partial w} + u_{II}\frac{\partial I}{\partial w}\right) \tag{1}$$

$$p\left(\frac{\partial M}{\partial w}f_{xM} + \frac{\partial x}{\partial w}f_{xx}\right) = 1 \tag{2}$$

where by differentiating constraints,

$$\frac{\partial I}{\partial w} = (pf_x - w)\frac{\partial x}{\partial w} + pf_M\frac{\partial M}{\partial w} - x = pf_M\frac{\partial M}{\partial w} - x \tag{3}$$

$$\frac{\partial f_M}{\partial w} = f_{MM}\frac{\partial M}{\partial w} + f_{Mx}\frac{\partial x}{\partial w}, \tag{4}$$

Essentially equations 1 and 2 are a system of equations with two unknowns $\partial M/\partial w$ and $\partial x/\partial w$; we solve them by expressing $\partial M/\partial w$ in terms of $\partial x/\partial w$ first and get $\partial x/\partial w$.

Plugging 3 and 4 into 1,

$$\begin{aligned}
(pf_M u_{IL} - u_{LL})\frac{\partial M}{\partial w} &= pu_I \frac{\partial f_M}{\partial w} + (pf_M u_{II} - u_{LI})\frac{\partial I}{\partial w} \\
(pf_M u_{IL} - u_{LL})\frac{\partial M}{\partial w} &= pu_I\left(f_{MM}\frac{\partial M}{\partial w} + f_{Mx}\frac{\partial x}{\partial w}\right) + (pf_M u_{II} - u_{LI})\left(pf_M \frac{\partial M}{\partial w} - x\right) \\
\frac{\partial M}{\partial w} &= \frac{pu_I f_{Mx}\frac{\partial x}{\partial w} - (pf_M u_{II} - u_{LI})x}{(pf_M u_{IL} - u_{LL}) - (pf_M u_{II} - u_{LI})pf_M - pu_I f_{MM}} \\
\frac{\partial M}{\partial w} &= \frac{pu_I f_{Mx}\frac{\partial x}{\partial w} + (u_{LI} - pf_M u_{II})x}{2pf_M u_{IL} - u_{LL} - (pf_M)^2 u_{II} - pu_I f_{MM}}.
\end{aligned}$$

Since u is assumed to be quasiconcave,

$$u_{LL} - 2pf_M u_{IL} + (pf_M)^2 u_{II} = \begin{pmatrix} pf_M \\ -1 \end{pmatrix} \cdot \begin{pmatrix} u_{II} & u_{IL} \\ u_{LI} & u_{LL} \end{pmatrix} \begin{pmatrix} pf_M \\ -1 \end{pmatrix} < 0,$$

the denominator which we denote by A from now on is positive. Next plug $\partial M/\partial w$ into 2,

$$\begin{aligned}
\frac{\partial M}{\partial w} f_{xM} + \frac{\partial x}{\partial w} f_{xx} &= \frac{1}{p} \\
\frac{1}{A}\left[pu_I f_{Mx}\frac{\partial x}{\partial w} + (u_{LI} - pf_M u_{II})x\right] f_{xM} + \frac{\partial x}{\partial w} f_{xx} &= \frac{1}{p} \\
\frac{\partial x}{\partial w}\left(pu_I f_{Mx}^2 + Af_{xx}\right) + (u_{LI} - pf_M u_{II})xf_{Mx} &= \frac{A}{p}
\end{aligned}$$

Therefore,

$$\frac{\partial x}{\partial w} = \frac{A/p - (u_{LI} - pf_M u_{II})xf_{Mx}}{pu_I f_{Mx}^2 + Af_{xx}}.$$

Denote the denominator

$$B = pu_I f_{Mx}^2 + Af_{xx} = pu_I\left(f_{Mx}^2 - f_{xx}f_{MM}\right) - f_{xx}\left(u_{LL} - 2pf_M u_{IL} + (pf_M)^2 u_{II}\right) < 0.$$

Simplify more,

$$\begin{aligned}
\frac{\partial x}{\partial w} &= \frac{1}{p}\frac{A}{B} - \frac{1}{B}(u_{LI} - pf_M u_{II})xf_{Mx} \\
&= \frac{1}{p}\frac{A}{B} - \frac{1}{B}\frac{1}{u_I}(u_I u_{LI} - u_L u_{II})xf_{Mx}.
\end{aligned}$$

Since $A > 0$ and $B < 0$, the first part is negative, the substitution effect. And since leisure is normal good,

$$u_I u_{LI} - u_L u_{II} \geq 0.$$

2. Provide conditions under which a rise in w necessarily reduces D's demand for aides.

   Solution. *If the second term is non-positive then $\partial x/\partial w$ is negative. The sufficient condition is either to have $f_{Mx} \geq 0$ or to have $u_I u_{LI} - u_L u_{II} = 0$. When the utility function is quasilinear in income, that is*

$$\begin{aligned}
u(L,I) &= h(\phi(L) + I), \\
u_{LI}u_I - u_L u_{II} &= \left[h''\phi'(L)\right]h' - \left[h'\phi'(L)\right]h'' = 0.
\end{aligned}$$

(Feel free to assume that all solutions are interior and unique, and that the customary calculus assumptions needed to assure this are satisfied.)