Math Facts for the Second Welfare Theorem

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Definition 1. **Minkowski sumset** of $S \in \mathbb{R}^L$ and $T \in \mathbb{R}^L$ is defined as

$$S + T \equiv \{s + t : s \in S, t \in T\}$$

Example 2. If $S = [0,1]$ and $T = (5,6)$, then $S + T = (5,7)$.

Theorem 3. *S and T convex $\Rightarrow S + T$ convex.*

Proof. Take $s_i, t_i \in S, T, \alpha (s_1 + t_1) + (1-\alpha)(s_2 + t_2) = \alpha s_1 + (1-\alpha) s_2 + \alpha t_1 + (1-\alpha) t_2 \in S + T$. □

Theorem 4. *$\overline{S} + \overline{T} \subset \overline{S+T}$ where $\overline{X}$ is the closure of X including all the elements and limit points of X.*

Proof. If $s \in S$ and $t \in T$, it is trivial to show that $s + t \in S + T$. If $s^* \notin S$ but there exist $\{s_n\} \in S$ such that $s_n \to s^*$ then for $t \in T$, $s_n + t \to s^* + t$ and $s^* + t \in \overline{S+T}$. Similarly for $t^* \in \overline{T}$, $s + t^* \in \overline{S+T}$ for $s^* \in \overline{S}$. □

Remark 5. The converse ($\overline{S} + \overline{T} \supset \overline{S+T}$) may not be true.

Theorem 6 (Separating Hyperplane Theorem)**.** *Given two disjoint convex sets S and T, $\exists \mathbf{p} \neq 0$ such that*

$$\mathbf{p} \cdot s \geq \mathbf{p} \cdot t \quad \forall s \in S, t \in T.$$

We prove the Theorem through two lemmas.

Lemma 7 (Simple Separating Hyperplane Theorem)**.** *Given a closed and convex set S and $t \notin S$, $\exists \mathbf{p} \neq 0$ and a value $c \in \mathbb{R}$ such that $\mathbf{p} \cdot t > c$ and $\mathbf{p} \cdot s < c$ for every $s \in S$.*

Proof. Denote the point closest to t by $s \in S$. Let $\mathbf{p} = t - s$ and $c' = \mathbf{p} \cdot s$. First,

$$\mathbf{p} \cdot t - c' = \mathbf{p} \cdot t - \mathbf{p} \cdot s = \mathbf{p} \cdot (t - s) = \|t - s\|^2 > 0.$$

Second, for any $s' \in S$, $s' - s$ cannot make an acute angle with $t - s$, that is

$$\left(s' - s\right) \cdot (t - s) = \|s' - s\| \|t - s\| \cos\theta \leq 0,$$

that is

$$\mathbf{p} \cdot s' - \mathbf{p} \cdot s = \mathbf{p} \cdot s' - c' \leq 0.$$

Now take $c = c' + \varepsilon$ such that $\mathbf{p} \cdot t > c$, and $\mathbf{p} \cdot t > c > \mathbf{p} \cdot s$. □

Lemma 8 (Simple Supporting Hyperplane Theorem)**.** *Suppose S is convex and t is not an element of the interior of S. Then there exists $\mathbf{p} \neq 0$ such that $\mathbf{p} \cdot t \geq \mathbf{p} \cdot s$ for every $s \in S$.*

Proof. Take a sequence of $\{t^m\}$ such that each $t^m \notin \overline{S}$, by the simple Separating Hyperplane Theorem, there is some $\mathbf{p}^m \neq 0$ and c^m such that

$$\mathbf{p}^m \cdot t^m > c^m \geq \mathbf{p}^m \cdot s \quad \forall s \in S.$$

By taking the limit, since $t^m \to t$ and $c^m \to c$,

$$\mathbf{p} \cdot t \geq c \geq \mathbf{p} \cdot s.$$

□

The proof is broken down into two cases in each a lemma is used.

Proof to Theorem 6. Define $X = S - T$, and take the closure $\overline{X}$. There are two cases.

1. $0 \notin \overline{X}$. Then by the Simple Separating Hyperplane Theorem, there is some $\mathbf{p}$ such that

$$\mathbf{p} \cdot (s - t) \geq \mathbf{p} \cdot 0 = 0,$$

   or $\mathbf{p} \cdot s \geq \mathbf{p} \cdot t \ \forall s \in S, \forall t \in T$.

2. $0 \in \overline{X}$, but it cannot be an interior point of $\overline{X}$ as S and T are disjoint. Then by the Simple Supporting Hyperplane Theorem, there is $\mathbf{p} \neq 0$ such that $\mathbf{p} \cdot (s - t) \geq \mathbf{p} \cdot 0 = 0$.

□

Definition 9. $\succeq$ on X is **continuous** if it is preserved under limits: for any sequence of pairs $\{x^n, y^n\}_{n=1}^{\infty}$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succeq y$.

Definition 10. $\succeq$ satisfies **local non-satiation (l.n.s.)** on consumption set X if for every $x \in X$ and every $\varepsilon > 0$, there is an $x' \in X$ such that $\|x' - x\| \leq \varepsilon$ and $x' \succ x$.

Theorem 11. *If $\succeq$ is continuous, then $\succeq (x) \supset \overline{\succ (x)}$. If $\succeq$ satisfies local non-satiation, then $\succeq (x) \subset \overline{\succ (x)}$.*

Proof. If $y \succ x$, then $y \succeq x$ by definition. If $y \not\succ x$ but there exists $y^n \succ x$ and $y^n \to y$, then by continuity of $\succeq$, let $x^n = x$ for all n, then $y^n \succeq x^n$ implies that $y = \lim_n y^n \succeq \lim_n x^n = x$.

If $y \succ x$, then it is trivial. If $y \sim x$, then by l.n.s., for each n, there is some $y^n \succ y$ for $\|y^n - y\| < 1/n$, and $y^n \to y$. Since $y^n \succ x$, $y \in \overline{\succ (x)}$. □

Definition 12. For some $S \in \mathbb{R}^L$, **convex hull** of S is the smallest convex set containing S, or the intersection of all convex sets that contain S,

$$\text{Co } (S) = \cap \{C_\alpha : \text{convex } C_\alpha \supset S\}.$$

Or equivalently,

$$\text{Co } (S) = \left\{ \sum_j \alpha_j x_j : x_j \in S \forall j, \alpha_j \geq 0 \forall j, \sum \alpha_j = 1 \right\}.$$

Proposition 13. [1]*If S_1, S_2, S_3 are convex and open, then*

$$\text{Co } (\cup_i S_i) = \left\{ \sum_j \alpha_j s_j : s_j \in S_j, \alpha_j \geq 0, \sum \alpha_j = 1 \right\}.$$

Gerard Debreu and Herbert Scarf. A limit theorem on the core of an economy. *International Economic Review*, 4(3):235–246, 1963.

[1] useful for Debreu and Scarf [1963].